Autonomous equations and phase portrait;

Equations of the form $\frac{dy}{dt} = f(y)$ (where the independent variable is not directly involved) are called autonomous equations. One simple example: $\frac{dy}{dt} = ry$ where r is a constant. Indeed we know a great deal about it and about its cousin: $\frac{dy}{dt} = ry + k$.

Another, not very simple example is: $\frac{dy}{dt} = ly(P - y)$ where l and P are positive constants. This differential equation is called the logistic equation and is used in various population models. Now, as $ly(P - y)$, and its (partial) derivative with respect to y are both continuous everywhere Theorem 2.4.2 of your book ensures that every initial value problem $\frac{dy}{dt} = ly(P - y),\ y(t_0) = y_0$ does indeed have a unique solution. (Indeed, we can use separation of variables or treat the equation as a Bernoulli equation to solve most of these initial value problems.) But the quality of the solutions may change with the change of initial values. For this a qualitative study is in order.

Consider then $\frac{dy}{dt} = ly(P - y)$ (1)

Note that (1) has two equilibrium solutions: $y(t) = 0$ and $y(t) = P$. These do indeed correspond to the initial conditions $y(0) = 0$ and $y(0) = P$.

Next, note that the sign of $\frac{dy}{dt} = ly(P - y)$ depends upon the position of y as shown below:

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ly  - - -   -    -    -      -  | + + + + + + +  + + + + + +   + +
P - y + + + + + + + + + + ++ + + + + + + +| - - - - - - - - - -
________________________________|____________________|____________________________
                                0                    P
ly(P - y) - - -  - - - - - - - - | + + + + + + + +| - - - - - - - - --
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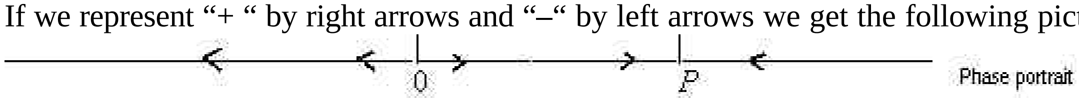
If we represent "+ " by right arrows and "–" by left arrows we get the following picture:



If the arrows indicate something moving in a certain direction then the above picture seems to indicate that everything close to the point P is moving towards P and everything close to 0 is moving away from 0. As we shall see below this is exactly what is happening: solutions that happen to have a negative value would always be negative valued and decreasing (going away from 0) and functions that have a value strictly between 0 and P would always have values strictly between 0 and P and would always be increasing (moving towards P and away from 0). Finally the solutions that have a value greater than P would always have values greater than P but will always be decreasing (coming down towards P). This diagram that tells the story of all the solutions is called the phase portrait of $\frac{dy}{dt} = ly(P-y)$. In phase portrait the point P seems to attract every function and so is called the attractor and the point 0 seems to repel everything and so is called a repeller. We justify our claims in the following.

Note that $\frac{dy}{dt} < 0$ if y happens to be less than 0. Thus for example if $y(0) < 0$ then the solution $y(t)$ to the initial value problem $\frac{dy}{dt} = ly(P-y),\ y(0) = y_0 < 0$ is ever decreasing.

Next, if y happens to be such that $0 < y < P$ then $\frac{dy}{dt} > 0$. Thus every solution of the initial value problem $\frac{dy}{dt} = ly(P-y),\ y(0) = y_0,\ 0 < y_0 < P$ is ever increasing as long as the y value is less than P. Now no variable solution can cross $y(t) = P$, because as soon as it does, it becomes the constant function $y(t) = P$. This means that $y(t) = P$ is the horizontal asymptote of every solution of the initial value problem

$$\frac{dy}{dt} = ly(P-y),\ y(0) = y_0,\ 0 < y_0 < P.$$

Finally if y happens to be greater than P then as the graph of y cannot ever cross $y(t) = P$ so y will always be greater than P but then $\frac{dy}{dt}$ will always be negative and so y will always be decreasing (towards P!). So $y(t) = P$ will be the horizontal asymptote of all the solutions of the initial value problem: $\frac{dy}{dt} = ly(P-y),\ y(0) = y_0,\ y_0 > P$. Now to support all this reasoning we give here a picture of the direction field of $\frac{dy}{dt} = ly(P-y)$ for $l = 1$ and $P = 1$.

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> with(DEtools):
DEplot(diff(y(t),t)=y(t)*(1-y(t)),y(t),t=-5..5,y(t)=-
1..1.5,[[y(0)=0],[y(0)=-.1],[y(-3)=1.4],[y(-1)=.5]]);
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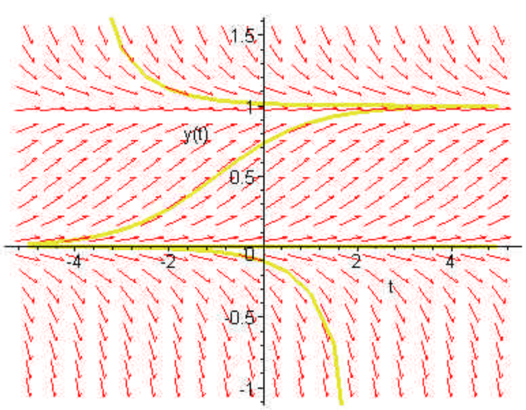
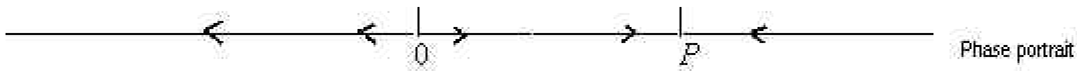


The phase portrait that is the picture below gives us in one dimension what the direction field gives us in a two dimensional picture.



You know the importance of phase portrait, now all you need is a little bit of practice with finding the phase portrait. Once you have practice with making the phase portrait you will be able to glean the direction field from it. For this let us have a couple of examples.

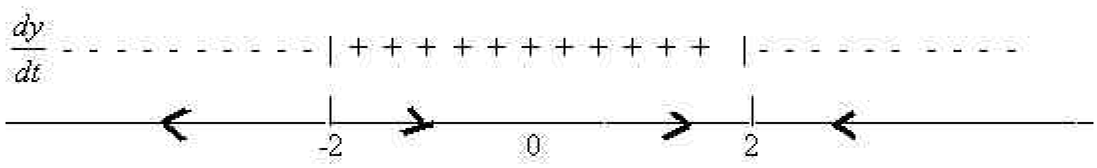
Example: Plot the phase portrait of $\frac{dy}{dt} = 7(4 - y^2)$.

Sol. We have $\frac{dy}{dt} = 7(2 - y)(2 + y)$

$\frac{dy}{dt}$ - - - - - - - - - - - | + + + + + + + + + + + |- - - - - - - - - - -

-2 0 2

So we have



Now based on this phase portrait you can complete the story the way you like at least saying that there are two equilibrium solutions: *y = -2* and *y=2* and of these *y=2* is an attractor. You can now make the “Deplot” to check your answer and to see what it really means.

Exercise: (i) Plot the phase portrait of $\frac{dy}{dt} = y(1 - y^2)$.

(ii) Plot the phase portrait of $\frac{dy}{dt} = ry$ for your choice of r with $|r| < 1$ and comment. Is $y = 0$ an attractor or a repeller?