QUESTION: (HD0802) I am interested in learning about the generalizations of Prüfer domains called \( v \)-domains and Prüfer \( v \)-multiplication domains, but they are studied using the star operations, which I am not very familiar with. Is there a way of defining these concepts without any mention of star operations?

ANSWER: You have raised an interesting and important question. Often other Mathematicians are not attracted to the notions of \( v \)-domain and its specializations because the jargon of star operations appears forbidding, though it is not. In the following I provide characterizations, of some of these domains. In my answer I prove statements, that, when used as definitions, do not mention any star operations. Note that “officially” an integral domain is a \( v \)-domain (resp., a Prüfer \( v \)-multiplication domain (PVMD)) if for every nonzero finitely generated (fractional) ideal \( A \) of \( D \) we have \((AA^{-1})_v = D\) (resp., \((AA^{-1})_t = D\)). So to understand the justification of the new definitions you will need some basic knowledge of the star operations. For this the best source available is sections 32 and 34 of Gilmer [2]. For now, if you are in a hurry look up HD0311 for star operations and note the following. The letter \( D \) denotes an integral domain with quotient field \( K \). By \( F(D) \) we denote the set of all nonzero fractional ideals of \( D \) and by \( f(D) \) the members of \( F(D) \) that are finitely generated. We use, for \( D \)-submodules \( A, B \) of \( K \) (such as the members of \( F(D) \)), the notation \( A : B \) to denote the set \( \{ x \in K : xB \subseteq A \} \) and denote \( D : A \) by \( A^{-1} \) which belongs to \( F(D) \) whenever \( A \) does. If \( A \subseteq B \) then \( A^{-1} \supseteq B^{-1} \), for \( A, B \in F(D) \). Moreover from the definition it follows that \((AA^{-1}) \subseteq D \) and \( D^{-1} = D \). Note that for \( A \in F(D) \), \( A_v = (A^{-1})^{-1} = D : (D : A) \) and that \( A_t = \cup (F_v : F \subseteq A \land F \in f(D)) \). It can be shown easily that \((A_v)^{-1} = A_v^{-1} \). If \( A \in F(D) \) is such that \( A = A_v \) (resp., \( A = A_t \)) we say that \( A \) is a \( v \)-ideal (resp., a \( t \)-ideal). A \( v \)-ideal is also called a divisorial ideal. \( A^{-1} \) is a \( v \)-ideal, and every invertible ideal is a \( v \)-ideal and a \( t \)-ideal. Also if there is a finitely generated fractional ideal \( B \) such that \( A = B_v \) we say that \( A \) is a \( v \)-ideal of finite type.

For a star operation \( * \) call \( A \in F(D) \) \(*\)-invertible if there is \( B \in F(D) \) such that \((AB*) = D \). It can be shown that in this case \((B*) = A^{-1} \). (You can take \( *, \) here, as a general name for the \( v \) - and \( t \)-operations and note that an invertible ideal is both \( v \)- and \( t \)-invertible.) So, \( D \) is a \( v \)-domain (resp., PVMD) if every \( A \in f(D) \) is \( v \)-invertible (resp., \( t \)-invertible), as indicated earlier. It can be shown that \( A \in f(D) \) is \( t \)-invertible if and only if \( A \) is \( v \)-invertible and \( A^{-1} \) is a \( v \)-ideal of finite type.

Lemma 1. An ideal \( A \in F(D) \) is \( v \)-invertible if and only if \( A^{-1} : A^{-1} = D \).

Proof. Suppose that \( A^{-1} : A^{-1} = D \). Let \( x \in (AA^{-1})^{-1} \supseteq D \). Then \( x(AA^{-1}) \subseteq D \) or \( xA^{-1} \subseteq A^{-1} \) or \( x \in A^{-1} : A^{-1} = D \). So \((AA^{-1})^{-1} \subseteq D \) and we have \((AA^{-1})^{-1} = D \). This gives \((AA^{-1})_v = D \).

Conversely if \( A \) is \( v \)-invertible then \((AA^{-1})^{-1} = D \). Let \( x \in A^{-1} : A^{-1} \supseteq D \). Then \( xA^{-1} \subseteq A^{-1} \). Multiplying both sides by \( A \) and applying the \( v \)-operation we get \( x \in D \). So, \( D \subseteq A^{-1} : A^{-1} \subseteq D \) and the equality follows.

Proposition 2. For an integral domain \( D \) the following are equivalent.

(0) \( D \) is a \( v \)-domain.
(1) \( A^{-1} : A^{-1} = D \) for each \( A \in f(D) \).
(2) \( A_v : A_v = D \) for each \( A \in f(D) \).
(3) \( A^{-1} : A^{-1} = D \) for each two generated \( A \in f(D) \).
(4) \( ((a) \cap (b)) : (a) \cap (b) = D \) for all \( a, b \in D \setminus \{0\} \).

Proof. (0) \( \Leftrightarrow \) (1) follows from Lemma 1 once we note that \( D \) is a \( v \)-domain if and only if each \( A \in f(D) \) is \( v \)-invertible.

(0) \( \Rightarrow \) (2). Let \( x \in A_v : A_v \supseteq D \). Then \( x A_v \subseteq A_v \). Multiplying both sides by \( A^{-1} \) and applying the \( v \)-operation we get \( x (A_v A^{-1})_v \subseteq (A_v A^{-1}) v \). But by \( (0) (A_v A^{-1})_v = D \) and so \( x D \subseteq D \). This forces \( D \subseteq A_v : A_v \subseteq D \).

(2) \( \Rightarrow \) (0). Let \( x \in (A_v A^{-1})^{-1} \supseteq D \). Then \( x (A_v A^{-1}) \subseteq D \). But then \( x A_v \subseteq A_v \), which gives \( x \in A_v : A_v = D \). But then \( D \subseteq (A_v A^{-1})^{-1} \subseteq D \), which means that every \( A \in f(D) \) is \( v \)-invertible.

(1) \( \Rightarrow \) (3) is obvious.

(3) \( \Rightarrow \) (4). Let \( a, b \in D \) and by (3) \( (a, b)^{-1} : (a, b)^{-1} = D \) or \( (a) \cap (b) : (a) \cap (b) = D \) which is the same as \( ((a) \cap (b)) : ((a) \cap (b)) = D \).

(4) \( \Rightarrow \) (0). Recall that \( D \) is a \( v \)-domain if and only if every two generated nonzero ideal of \( D \) is \( v \)-invertible [3]. Now let \( x \in ((a, b), (a, b)^{-1}) \supseteq D \), where \( a, b \in D \setminus \{0\} \). Then \( x (a, b), (a, b)^{-1} \subseteq D \). Or \( x (a, b)^{-1} \subseteq (a, b)^{-1} \). Or \( x (a, b) \subseteq (a, b)^{-1} \). Or \( x( (a) \cap (b)) \subseteq (a) \cap (b) \) or \( x \in ((a) \cap (b)) : ((a) \cap (b)) = D \). This forces \( D \subseteq ((a, b), (a, b)^{-1})^{-1} \subseteq D \).

An immediate consequence of the above Proposition is the following characterization of \( PVMD \)'s, which stems from the fact that \( D \) is a \( PVMD \) if and only if every two generated nonzero ideal of \( D \) is \( v \)-invertible [3] if and only if every two generated ideal \( (a, b) \) of \( D \) is \( v \)-invertible such that \( (a, b)^{-1} = (x_1, x_2, ..., x_r) \), where \( r \in N \). Note that \( (a, b)^{-1} = (x_1, x_2, ..., x_r) \) implies that \( (a) \cap (b) = (x_1, x_2, ..., x_r) \). Multiplying both sides by \( ab \) and using the definition of the \( v \)-operation we have \( (a) \cap (b) = (ab x_1, ab x_2, ..., ab x_r) \). Recall from the introduction that a fractional \( v \)-ideal \( A \) is said to be a \( v \)-ideal of finite type if there exist \( a_1, a_2, ..., a_n \in A \) such that \( A = (a_1, a_2, ..., a_n) \). Call an integral domain \( D \) a \( v \)-finite conductor (\( v \)-FC-) domain if \( (a) \cap (b) \) is a \( v \)-ideal of finite type, for every pair \( a, b \in D \setminus \{0\} \).

You may say that in the above definition of \( v \)-FC-domains there is still a mention of the \( v \)-operation. We have a somewhat contrived solution for this, in the form of the following characterization of \( v \)-FC-domains.

Proposition 3. An integral domain \( D \) is a \( v \)-FC-domain if and only if for each pair \( a, b \in D \setminus \{0\} \) there exist \( y_1, y_2, ..., y_n \in K \setminus \{0\} \) such that \( (a, b)_v = \bigcap y_i D \). Consequently \( D \) is a \( v \)-FC-domain if and only if for each pair \( a, b \in D \setminus \{0\} \) there exist \( y_1, y_2, ..., y_n \in K \setminus \{0\} \) such that \( (a) \cap (b)^{-1} = \bigcap y_i D \).

Proof. Let \( D \) be a \( v \)-FC-domain and let \( a, b \in D \setminus \{0\} \). Then there are \( a_1, a_2, ..., a_n \) such that \( (a) \cap (b) = (a_1, a_2, ..., a_n) \). Dividing both sides by \( ab \) we get \( (a) \cap (b) = (a_1/ab, a_2/ab, ..., a_n/ab) \). But \( (a_1/ab, a_2/ab, ..., a_n/ab) = (a_1/ab, a_2/ab, ..., a_n/ab) \). This gives \( (a) \cap (b) = (a_1/ab, a_2/ab, ..., a_n/ab) \). Conversely if \( (a, b)_v = \bigcap y_i D \) then \( (a) \cap (b)^{-1} = (a, b)_v^{-1} = \bigcap y_i D^{-1} \). This gives \( (a) \cap (b)_v = (a, b)^{-1} = \bigcap y_i D \). For the "consequently" part note that \( ((a) \cap (b))^{-1} =
Corollary 4. For an integral domain $D$ the following are equivalent.

1. $D$ is a PVMD
2. for all $a, b \in D \setminus \{0\}$ we have $((a) \cap (b))^{-1}$ a finite intersection of principal fractional ideals and $((a) \cap (b)) : ((a) \cap (b)) = D$
3. $D$ is a $v$-FC-domain and for all $a, b \in D \setminus \{0\}$ we have $((a) \cap (b)) : ((a) \cap (b)) = D$.

Recall that $D$ is called a finite conductor (FC-) domain if $((a) \cap (b))$ is finitely generated for each pair $a, b \in D$. Just to show how far we have traveled since [9] we state and prove the following corollary.

Corollary 5. An integrally closed FC-domain is a PVMD.

Proof. We first note that since $D$ is integrally closed $A = D$ for every finitely generated ideal $A$ of $D$. So for each pair $a, b \in D \setminus \{0\}$, since $D$ is FC, $((a) \cap (b)) : ((a) \cap (b)) = D$. But this makes $D$ a $v$-domain, by Proposition 2 and a PVMD by Corollary 4.

While the result in Corollary 4 was the main result of [9] that paper contained many useful techniques. For example [9] was the first in the literature to introduce the formula $(AD)^{s}_{v} = (A_{v}D)^{s}_{v}$ where $A$ is a finitely generated (nonzero) ideal of $D$ and $S$ is a multiplicative set of $D$, among other useful techniques.

Lemma 1 can also be instrumental in characterizing completely integrally closed (CIC-) domains. There is enough information for CIC-domains in section 34 of [2]. Also the approach in this answer leads to a characterization of Krull domains in a manner similar to the characterization of $v$-domains leading to the characterization of PVMD's.

Proposition 6. For an integral domain $D$ the following are equivalent:

1. $D$ is a CIC-domain
2. $A^{-1} : A^{-1} = D$ for all $A \in F(D)$

For the proof note that $D$ is CIC if and only if every $A \in F(D)$ is $v$-invertible [2, Proposition 34.2, Theorem 34.3] and (2) provides precisely that by Lemma 1.

Proposition 7. The following are equivalent for an integral domain:

1. $D$ is a Krull domain;
2. for each $A \in F(D)$ there exist $y_{1}, y_{2}, ..., y_{n} \in A$ such that $A^{-1} = \cap_{y_{i}}^{1}, D$

and for all $a, b \in D \setminus \{0\}$, $((a) \cap (b)) : ((a) \cap (b)) = D$;
3. for each $A \in F(D)$ there exist $x, y \in A$ such that $A^{-1} = \frac{1}{x} D \cap \frac{1}{y} D$ and for all $a, b \in D \setminus \{0\}$, $((a) \cap (b)) : ((a) \cap (b)) = D$.

Before we prove Proposition 7 it seems pertinent to give some introduction. An integral domain $D$ is called a Mori domain if $D$ satisfies ACC on integral divisorial ideals. Different aspects of Mori domains were studied by Toshio Nishimura in a series of papers. For instance in [8] he showed that a domain $D$ is a Krull domain if and only if $D$ is a Mori domain and completely integrally closed. An integral domain $D$ is a Mori domain if and only if for each $A \in F(D)$ there exist $y_{1}, y_{2}, ..., y_{n} \in A \setminus \{0\}$ such that $A_{v} = (y_{1}, y_{2}, ..., y_{n})_{v}$ [7, Lemma 1], and it is easy to see that this characterization is equivalent to for each $A \in F(D)$
there exist \( y_1, y_2, \ldots, y_n \in A \setminus \{0\} \) such that \( A^{-1} = \cap_{y_i} D^* \). For a quick review of Krull domains the reader may consult the first few pages of [1]. A number of characterizations of Krull domains can be found in [4, Theorem 2.3]. The one that we can use here is: \( D \) is a Krull domain if and only if each \( A \in F(D) \) is \( t \)-invertible. Which means that \( D \) is a Krull domain if and only if for each \( A \in F(D) \), \( A \) is \( v \)-invertible and \( A^{-1} \) is of finite type. For another proof of a Mori domain being a Krull domain see [11, Corollary 2.2].

Proof. From the above discussion (1) \( \iff \) (2). Next (3) \( \Rightarrow \) (1) follows from (2)\( \iff \) (1). For (1) \( \Rightarrow \) (3) note that for all \( a, b \in D \setminus \{0\} \), \( ((a) \cap (b)) : ((a) \cap (b)) = D \), because of (1) \( \iff \) (2). For the remaining part recall from [5, Proposition 1.3] that if \( D \) is a Krull domain then for every \( A \in F(D) \) there exist \( x, y \in A \) such that \( A_v = (x, y)_v \).

Remark 8. In (2) of Proposition 7 we cannot say that for every \( A \in F(D) \) the inverse \( A^{-1} \) is expressible as a finite intersection of principal fractional ideals, because this would be equivalent to \( A_v \) being of finite type for each \( A \in F(D) \). But there do exist non-Mori domains \( D \) such that \( A_v \) is of finite type for all \( A \in F(D) \). For a discussion of those examples you may consult [10, section 2].

Remark 9. I hope that you realize that while the above results provide definitions of those concepts in terms of mainstream algebra they also indicate the importance of star operations as a means of getting deeper than where the mainstream techniques could not help.

Remark 10. I am grateful to Said El Baghdadi for reading an earlier version of the answer and catching a really bad error. David Dobbs, Marco Fontana and Evan Houston also helped.

References


