QUESTION: (HD0806) I was studying your paper "Factorization of certain sets of polynomials in an integral domain". In Theorem 5 of the paper for the proof of (1) <–> (2) you are quoting the reference of Arnold and Gilmer's paper, "on the contents of polynomials". But this paper does not contain the proof of the Result: Let D is an integral domain with identity having quotient field K. Then (1) If D is a Schreier ring, then for any positive integer n, D [X1...Xn] is inert in K [X1...Xn]. (2) If D [X1...Xn] is inert in K [X1...Xn] for some n greater than or equal to 1, then D is a Schreier ring. For that the author is writing that referee has communicated to the author. Sir do you have the proof of above result. If you have, then please send me.

ANSWER: The paper that you are referring to as mine is by Daniel Anderson, Pramod Sharma, and myself [ASZ, Focus on Commutative Rings Research, 153-156, Nova Sci. Publ., New York, 2006]. (You may find the preprint at http://www.lohar.com/researchpdf/factsepo.pdf) The other paper you refer to as Arnold and Gilmer's paper is [AG, Proc. Amer. Math. Soc. 24(1970), 556-562]. I would assume that you and any interested reader is familiar with the terminology used in the above papers.

It appears that you need the proof of $1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 1$ in Theorem 5 and that you agree with $1 \Leftrightarrow 3$. So if I show that $1 \Rightarrow 2 \Rightarrow 3$ we are done, but I will complete the whole proof using a trick that I have learned from a paper by David E. Rush [R, Mathematika 50 (2003), no. 1-2, 103–112 (2005)].

It is well known that D is a Schreier domain if and only if $D[X]$ is Schreier (see Cohn's paper [Proc. Cambridge Philos. Soc. 64(1968), 251-264])

$1 \Rightarrow 2$. Suppose that D is a Schreier domain and let $f(X) \in D[X]$ be nonconstant such that $f(X) = g(X)h(X)$ where $g(X), h(X) \in K[X]$. Then for some $r, s \in D\backslash\{0\}$ we have $rg(X), sh(X) \in D[X]$. So we have $rsf(X) = rg(X)sh(X)$. Since $D[X]$ is Schreier and $f(X) \mid rg(X)sh(X)$ in $D[X]$ we have $f(X) = u(X)v(X)$ such that $u(X) \mid rg(X)$ and $v(X) \mid sh(X)$ in $D[X]$. This gives $rg(X) = p(X)u(X)$ and $sh(X) = q(X)v(X)$. Thus we have $rsu(X)v(X) = p(X)u(X)q(X)v(X)$ or on cancellation we have $rs = p(X)q(X)$. Since this equation is in $D[X]$, $p(X), q(X)$ must be of degree 0 and hence constants. Denote $p(X)$ by p and $q(X)$ by q. Then $rs = pq$ and so $\frac{rs}{pq} = 1$. Take $u = \frac{r}{p}$ and $u^{-1} = \frac{s}{q}$ (see Definition 1 in [ASZ]).

$2 \Rightarrow 3$. Obvious because property (P) requires that if $f = gh$ in $K[X]$ with g, h non constant then $f = \alpha\beta$ in $D[X]$ α, β nonconstant and 2 provides $\alpha = ug(X), \beta = u^{-1}h(X)$ in $D[X]$.

$3 \Rightarrow 1$. Using Theorem 3, D is integrally closed. Now let $a, m, n \in D\backslash\{0\}$ such that $a \mid mn$ and set up $f(X) = aX^2 + (m+n)X + \frac{mn}{a} \in D[X]$. Rewriting $f(X) = a(X^2 + \frac{m+n}{a}X + \frac{mn}{a^2}) = a(X^2 + (\frac{m}{a} + \frac{n}{a})X + \frac{mn}{a^2}) = a(X + \frac{m}{a})(X + \frac{n}{a})$ in $K[X]$. By 3, $aX^2 + (m+n)X + \frac{mn}{a} = (cX + t)(dX + u)$ where $c, d, t, u \in D$. So, $aX^2 + (m+n)X + \frac{mn}{a} = cdX^2 + (cu + dt)X + tu$. Comparing coefficients we have $a = cd$, $m + n = cu + dt$ and $tu = \frac{mn}{a}$. The last of these equations gives $atu = mn$ or $cdtu = mn$. Set $cu = w$ and $dt = z$. Then we have $w + z = m + n$ and $wz = mn$. From the first of these take $m = w + z - n$ and substitute for m in $wz = mn$ to get $wz = (w + z - n)n$. Simplifying you get $n^2 - (w + z)n + wz = 0$.

Factorizing we get $(n-w)(n-z)=0$ which implies that $n=w$ or $n=z$. Take $n=z$, then $n=dt$ and substituting for n in $cdtu=mn$, we get $cu=m$. But then $a=cd$ where $c \mid m$ and $d \mid n$. If you take $n=w$ you will get $n=cu$ and $m=dt$. From both we see that for each triplet $a, m, n \in D \backslash \{0\}$ if $a \mid mn$ then $a=rs$ where $r \mid m$ and $s \mid n$. This property, along with D being integrally closed, implies that D is Schreier.

Do not forget to read that paper by David Rush. The title of the paper is "Quadratic Polynomials, factorization in integral domains and Schreier domains from pullbacks" and it appeared in Mathematika a journal published by University College London.