QUESTION (HD0303). How do you show that a one-dimensional Bezout domain is completely integrally closed?

To understand the answer fully you need to know the following:

(a). For any integral domain D we have $D = \bigcap D_M$ where M ranges over maximal ideals of D. A standard result.

(b) An integral domain $D,$ with quotient field $K,$ is called completely integrally closed (in K) if for $r \in K$, $r^n s \in D$ for all natural numbers n and some $s \in D\backslash\{0\},$ implies that $r \in D.$(There are other (equivalent) definitions and for those you can check section 13 of Gilmer's book, Multiplicative ideal theory, Marcel Dekker, 1972)

The simplest example of a completely integrally closed domain is a one dimensional valuation domain $(V,M).$ Because V is one dimensional local, for every pair r,s of nonzero nonunits of V, $r \mid s^m$ for some natural number m and $s \mid r^n$ for some natural number n (Theorem 108 of Kaplansky's book, Commutative rings, Allyn and Bacon 1970). Let K be the quotient field of $V.$ (V,M) is completely integrally closed because if you take $r \in K.$ Then $r \in V$ or $\frac{1}{r} \in V.$ If for some $s \in V\backslash\{0\}, r^n s \in V$ for all natural $n,$ then we claim that $r \in V,$ for if not and say $r \notin V$ then $\frac{1}{r} \in V.$ In this case $r = \frac{1}{v}$ for some nonzero nonunit $v \in V.$ But then $(\frac{1}{v})^n s \in V$ for all natural n which is impossible because $s \mid v^m$ for some m, so $(\frac{1}{v})^n s = (\frac{1}{v})^{n-m}(\frac{s}{v^m}) \in V$ implies that $1 \in v^{n-m}(\frac{v^m}{s})V$ for $n > m$ but v is a nonzero nonunit in V and $(\frac{v^m}{s}) \in V.$

(c) If $D = \bigcap R_\alpha$ where each R_α is completely integrally closed in the quotient field K of D then D is completely integrally closed in $K.$

Proof. Let $r \in K$ and let $s \in D\backslash\{0\}$ and suppose that $r^n s \in D$ for all natural $n.$ Then since $x \in D$ implies that $x \in R_\alpha$ for each $\alpha.$ So, $s \in R_\alpha$ for each α and for each α we have $r^n s \in R_\alpha$ for all natural $n.$ But then for each $\alpha,$ $r \in R_\alpha$ which means that $r \in \bigcap R_\alpha = D.$

(d) If D is a Bezout domain then for every prime ideal P of D, D_P is a valuation domain. (See P.M. Cohn's, "Bezout rings and their subrings", Proc. Cambridge Philos. Soc.

(e) If P is of height 1 in a Bezout domain D then D_P is a 1-dimensional valuation domain and hence by (b) D_P is a completely integrally closed integral domain.

Now if D is a one-dimensional Bezout domain, then every maximal ideal of D is of height one. So $D = \bigcap D_P$ where P ranges over height one primes. By (e) each D_P is a one-dimensional valuation domain and by (b) each D_P is completely integrally closed and by (c) $D = \bigcap D_P$ is completely integrally closed.

Comments:

1. David Anderson's comment about (d): Alternatively, note that a Bezout domain is Prufer (every finitely generated ideal is invertible) and use Theorem 64 of Kaplansky, Commutative rings, Allyn and Bacon 1970.