QUESTION (HD0310): If D is completely integrally closed then how is D integrally closed?

ANSWER: That a "comletely integrally closed is integrally closed" follows from the general fact that “integral" implies “almost integral”. You can read on integral elements and almost integral elements from sections 9 through 13 of [Gilmer, Multiplicative ideal theory, Marcel Dekker, 1972]. As your question is related to integral domains. I assume that by “D is (completely) integrally closed" you mean D is (completely) integrally closed in the field of fractions K of D.

In this set up D is completely integrally closed in K if for $x \in K$ such that there is a nonzero element $d \in D$ with $dx^n \in D$ for all natural numbers $n,$ we have $x \in D.$ On the other hand D is integrally closed if for $x \in K$ such that there are $a_0, a_1 \ldots a_{n-1} \in D$ with $x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0 = 0$ we have $x \in D.$

Now let D be completely integrally closed in K and let x be integral over $D.$ (We shall show that $x \in D.$) Now x being integral over D means that $x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0 = 0$ for some $n.$

This gives $x^n = -a_{n-1}x^{n-1} - \ldots - a_1 x - a_0$ ––––––––––(A)

From (A) it follows that $x^n \in Dx^{n-1} + \ldots + Dx^2 + Dx + D.$

We can use induction, and (A) to show that $x^r \in Dx^{n-1} + \ldots + Dx^2 + Dx + D$ for all natural numbers $r.$

$[[$ We already know that $x^i \in Dx^{n-1} + \ldots + Dx^2 + Dx + D$ for $0 \leq i \leq n.$ (Taking $x^0 = 1$) We complete the task by showing that $S_k : x^{n+k} \in Dx^{n-1} + \ldots + Dx^2 + Dx + D$ is true for all $k \geq 0.$ Now S_k is true for $k = 0,$ as we have already seen in (A). Suppose S_k true for some $k \geq 0.$ Then $x^{n+k} \in Dx^{n-1} + \ldots + Dx^2 + Dx + D.$

That is $x^{n+k} = d_{n-1}x^{n-1} + \ldots + d_2 x^2 + d_1 x + d_0$ –––––––(B)

for $d_{n-1}, \ldots, d_2, d_1, d_0 \in D.$

Multiply both sides of (B) by x to get

$x^{n+k+1} = d_{n-1}x^n + \ldots + d_2 x^3 + d_1 x^2 + d_0 x$ –––––––––––––(C)

Substitute the value of x^n from (A) in the right hand side of (C) to get

$x^{n+k+1} = d_{n-1}(-a_{n-1}x^{n-1} - \ldots - a_1 x - a_0) + \ldots + d_2 x^3 + d_1 x^2 + d_0 x.$

Simplify and organize as a sum of powers of x to get an expression like.

$x^{n+k+1} = u_{n-1}x^{n-1} + \ldots + u_2 x^2 + u_1 x + u_0$ where, for instance, $u_{n-1} = -d_{n-1}a_{n-1}$ and $u_0 = -d_{n-1}a_0.$

But this means that S_{k+1} is true. So, we have shown that S_0 is true and S_k true implies that S_{k+1} is true and so by induction S_k is true for all integers $k \geq 0.$ But this means that $x^r \in Dx^{n-1} + \ldots + Dx^2 + Dx + D$ for all integers $r \geq 0.]]$

Now as $x \in K$ we can write $x = \frac{u}{v},$ where $v \neq 0$, $u \in D.$ Now, if we put $d = v^n \in D$ we find that $dx^i \in D$ for all i with $0 \leq i \leq n.$ But then $dDx^i \subseteq D$ for all i with $0 \leq i \leq n.$ Consequently from $x^r \in Dx^{n-1} + \ldots + Dx^2 + Dx + D$ we get $dx^r \in dDx^{n-1} + \ldots + dDx^2 + dDx + dD \subseteq D.$ Hence for each nonnegative integer r we have $dx^r \in D.$ Because D is completely integrally closed this means that $x \in D.$ Since x is an arbitrary element of K that we assumed was integral over D we conclude that D is integrally closed.

Note. The material within square brackets [[ ]], is the proof by induction. If you do not need it skip it.

Comments:

1. (David Anderson): It would be easier to see that integral implies almost integral if you know that x is integral over D if and only if $x \in I : I$ for some finitely generated fractional ideal I of D and that x is almost integral over D if and only if $x \in I : I$ for some fractional ideal I. (Here $I : I = \{x \in K : xI \subseteq I\}$)