QUESTION:(HD0501) In Huneke's book "Tight Closure and Its Applications", he mentioned the following fact regarding *complete integral closure* (pg. 14, Example 1.6.1): Let R be a Noetherian. integral domain with fraction field K.

Let α be an element in K. If there is a nonzero $c \in R$ such that $c(\alpha^n) \in R$ for **infinitely many** n, then α is integral over R.

I couldn't figure out how to show this, although I understand why this is true when "infinitely many" is replaced by "all", which

is the definition of *almost integral*.

ANSWER I: Let $\alpha \in K\backslash\{0\}$ and c the element such that $c\alpha^n$ is in R for infinitely many values of n. Consider the ideal of R:

$I = (c\alpha^{n_1}, c\alpha^{n_2}, c\alpha^{n_3}...)$ where $n_1 < n_2 < n_3 <$ ...are the increasing exponents such that $c\alpha^{n_i} \in R$.

Since R is Noetherian, this ideal can be generated by a finite list of the above. That is:

$I = (c\alpha^{m_1}, c\alpha^{m_2}, ...c\alpha^{m_r})$, where $m_1 < m_2 < ... < m_r$

Let s be such that $ca^s \in R$ and $s > m_r$. Then we have that for $r_1, r_2, ..., r_m \in R$, $c\alpha^s = r_1 c\alpha^{m_1} + r_2 c\alpha^{m_2} + ... + r_m c\alpha^{m_r}$

Divide both sides above by c to get the equation of integrality.

There is an alternative way of seeing Huneke's statement if you know that the integral closure of a Noetherian domain is a Krull domain and that a Krull domain is an intersection of discrete rank one valuation domains. In fact we have a more general result. It follows from **HD0303** that an intersection of completely integrally closed domains is completely integrally closed and that a rank one valuation domain is completely integrally closed. Also recall from **HD0310** that a completely integrally closed domain is integrally closed. Let us call $\alpha \in K\backslash\{0\}$ pseudo almost integral over R if there is a $c \in R\backslash\{0\}$ such that $c\alpha^n \in R$ for infinitely many n.

Theorem 1. Let R be an integral domain with fraction field K and with integral closure $R^/$ completely integrally closed. Let α be an element in K. If α is pseudo almost integral over R then α is integral over R.

Proof. We show that α is almost integral over R. For this we show that for any natural number m we have $c\alpha^m \in R^/$. Now since $c\alpha^n \in R$ for infinitely many n we can choose $n > m$. As $c \in R$ we have $c^{(n-m)} \in R$, also as $c\alpha^n \in R$ we have $(c\alpha^n)^m \in R$. So $c^{(n-m)}(c\alpha^n)^m \in R$. Simplifying, we have $c^n\alpha^{nm} = (c\alpha^m)^n \in R$. This forces $c\alpha^m \in R^/$ for each m. Now since $R^/$ is completely integrally closed we have $\alpha \in R^/$.

Corollary 2. (To the proof.) Let R be an integral domain with fraction field K and with complete integral closure R^* completely integrally closed. Let α be an element in K. If α is pseudo almost integral over R, then α is almost integral over R.

Proof. As in the proof of Theorem 1, we get $(c\alpha^m)^n \in R \subseteq R^*$. Being completely integrally closed R^* is integrally closed and so $(c\alpha^m)^n \in R^*$ implies that $c\alpha^m \in R^*$, for each m. Since $c \in R^*$, α is almost integral over R^*. But R^* being completely integrally closed we have $\alpha \in R^*$.

Corollary 3 (**ANSWER II**). Let R be an integral domain with fraction field K and with integral closure $R^/$ an intersection of rank one valuation domains. Let α be an element in K. If there is a nonzero $c \in R$ such that $c\alpha^n \in R$ for infinitely many n, then α is integral over R.

It is well known that the complete integral closure R^* of R may not be completely integrally closed, see Example 1 of Gilmer and Heinzer [J. Aust. Math. Soc. 6(1966), 351-361]. The following corollary gives an alternative way of testing whether the complete integral closure of R is completely integrally closed or not.

Corollary 4. Let R be an integral domain with fraction field K. If there is in K a pseudo almost integral element α such that α is not almost integral over R then R^* is not completely integrally closed.

There do exist non-Noetherian domains whose integral closures (complete integral closures) are expressible as intersections of rank one valuation domains. Here are some examples.

Example 5. Let Q be the field of rational numbers, $\widetilde{Q}$ the algebraic closure of Q and let X be an indeterminate. The subring D of $\widetilde{Q}[X]$ defined by $D = Q + X\widetilde{Q}[X] = \{f(X) \in \widetilde{Q}[X] : f(0) \in Q\}$ is non-Noetherian yet its integral closure is $\widetilde{Q}[X]$ a PID.

Illustration. Because $[\widetilde{Q} : Q]$ is infinite the ideal $X\widetilde{Q}[X]$ is infinitely generated and so D is not Noetherian. Next, because the quotient field of D is $\widetilde{Q}(X)$ and because $\widetilde{Q}$ is algebraic (and hence integral) over Q the integral closure of D contains $\widetilde{Q}[X]$, but $\widetilde{Q}[X]$ is integrally closed.

Example 6. Let $Q^+ = \{q \in Q : q \geq 0\}$ and let $S = \{X^q : q \in Q^+\}$. The semigroupring $\widetilde{Q}[S] = \{\sum a_i X^{q_i} : a_i \in \widetilde{Q}, q_i \in Q^+\}$ is known to be a one dimensional Bezout domain. Let (S) denote the ideal of $\widetilde{Q}[S]$ generated by S. Construct the ring $D = Q + (S)\widetilde{Q}[S] = \{f \in \widetilde{Q}[S] : f(0) \in Q\}$. Then the integral closure of D is $\widetilde{Q}[S]$ a one dimensional Bezout domain and hence an intersection of rank one valuation domains.

Example 7. The domain $Z + XQ[X]$ is integrally closed but its complete integral closure is $Q[X]$ which is a PID and hence an intersection of discrete rank one valuation domains.

We have seen in Corollary 4 that if there is $\alpha \in K$ such that α is pseudo almost integral over R but not almost integral over R then R^* is not integrally closed. This leads to the following question: Are there any pseudo almost integral elements that are not almost integral?

The answer is: Yes, there are. For completeness we include below an example from Hammann, Houston and Johnson [Pacific J. Math. 135(1)(1988), 65-79.]

Example 8. Let t, y be two indeterminates over a field F, let $D = F[t, \{ty^{2^n}\}_{n=0}^{\infty}]$. Then y is pseudo almost integral over D but not almost integral over D.

Illustration. From the definition y is pseudo almost integral. To show that y is not almost integral we need to show that for each $c \in F[t, \{ty^{2^n}\}_{n=0}^{\infty}] \backslash \{0\}$

there is a natural number n such that $cy^n \notin F[t, \{ty^{2^n}\}_{n=0}^{\infty}]$. Since each c is a sum of monomials in t and y (and since to be in $D\backslash(0)$ each summand has to be a monomial in t and ty^{2^n}) it is sufficient to consider the case when c is a monomial. In general a monomial in D is of the form $c = t^m \prod_{i=1}^{k}(ty^{2^{r_i}})^{m_i}$, where $r_1 > r_2 > ... > r_k > 0, m \geq 0$ and $m_i > 0$. Rewriting as a monomial in t and y we have $c = t^u y^v$ where $u = m + m_1 + ... + m_k$ and $v = \sum m_i 2^{r_i}$. We use the fact that u is constant. Choose n such that $n = (\sum_{i=1}^{u+1} 2^{s_i}) - v > 0$ where s_i are suitable positive integers with $s_1 > s_2 > ... > s_{u+1}$. So, $cy^n = t^u(y)^{(\sum_{i=1}^{u+1} 2^{s_i})}$. Now to be in D, cy^n must be in $R = F[t, \{ty^{2^n}\}_{n=0}^{w}]$ for some w and so for each power of y there must be a t attached. But there are only u of the t factors and $u+1$ of the $y^{2^{s_i}}$, forcing cy^n to have, by the pigeon hole principle, at least one factor of the form $ty^{(2^a+2^b+...)}$ where $a, b... \in \{s_i\}_{i=0}^{u+1}$. But as $a, b, ...$ are distinct and positive $ty^{(2^a+2^b+...)} \neq ty^{2^n}$ for $n = 0, 1, 2, ...w$ a contradiction.

Remark 9. It may be noted that in view of Corollary 4, the integral domain D of Example 8 is another example of a domain whose complete integral closure is not completely integrally closed.

Remark 10. For a systematic treatment of almost integrity in commutative rings see section 13 of Gilmer's "Multiplicative Ideal Theory", Marcel Dekker, 1972.

Remark 11. While an intersection of rank one valuation domains is completely integrally closed a completely integrally closed integral domain may not be expressible as an intersection of rank one valuation domains. To see this consult Nakayama's papers [Proc. Imp. Acad. Tokyo 18(1942), 185-187] and [Proc. Imp. Acad. Tokyo 18(1942), 233-236]. The example is a completely integrally closed Bezout domain D that has no height one prime ideals and so cannot be expressed as an intersection of rank one valuation domains. This example can also be used to show that a ring of fractions of a completely integrally closed integral domain may not be completely integrally closed.

(David Anderson, Jim Coykendall and Evan Houston helped, Zafrullah)

Daniel Anderson recommends that we should also note that there are proper subsets T of natural numbers such that if for a $y \in K$ there is a $c \in D\backslash(0)$ such that $cy^x \in D$ for each $x \in T$ then y is almost integral over D. Here are two examples of such subsets. Let us call

Proposition 12. If for some positive integer p we define $T_p = \{n \in N : n \geq p\}$ and if $y \in K$ is such that for some $c \in D\backslash(0)$, $cy^n \in D$ for all $n \in T_p$ then y is almost integral over D.

Proof. Set $y = \frac{a}{b}$. Then $c_1 = b^p c$ has the property that $c_1 y^n \in D$ for all natural numbers n.

Proposition 13. If $T = \{t_n = \alpha + (n-1)\delta\}_{n=1}^{\infty}$, where α, δ are positive integers and if $y \in K$ such that for some $c \in D\backslash(0)$, $cy^{t_n} \in D$, for all $n \in N$

then y is almost integral over D.

Proof. Set $y = \frac{a}{b}$ and set $\epsilon = \max(\alpha, \delta) - 1$.Then $c_1 = b^\epsilon c$ has the property that $c_1 y^n \in D$ for all natural numbers n. It seems fair to elaborate a little. The idea is that if $\alpha \geq \delta$ say then $\epsilon = \alpha - 1$, we have $b^{\alpha-1}c(y^i) \in D$ for $1 \leq i \leq \alpha - 1$, because of the $b^{\alpha-1}$ factor and as $cb^\alpha \in D$, by the given data we have covered y^i for $1 \leq i \leq a$. Now suppose we have shown that $b^{\alpha-1}c(y^i) \in D$ for all i with $1 \leq i \leq \alpha + (n-1)\delta$ and note that for $\alpha + (n-1)\delta < j < \alpha + n\delta$ we have $j = \alpha + (n-1)\delta + k$ such that $k = 1, 2, ...\delta - 1 \leq \alpha - 1$. So, $b^{\alpha-1}c(y^j) = b^{\alpha-1}c(y^{(\alpha+(n-1)\delta+k)}) = b^{\alpha-1}(cy^{(\alpha+(n-1)\delta)})y^k$. But $cy^{(\alpha+(n-1)\delta)} \in D$ and $b^{\alpha-1}y^k \in D$ bcause $k \leq \alpha - 1$. This means that $b^{\alpha-1}y^i \in D$ for every $1 \leq i < \alpha$ and $b^{\alpha-1}y^i \in D$ for all i with $t_n < i < t_{n+1}$ where n varies over N. Indeed this means that $b^{\alpha-1}y^i \in D$ for all $i \in N$. With minor adjustments the above procedure can be repeated for the case of $\max(\alpha, \delta) = \delta$.

For immediate corollaries we note that $y \in K$ is almost integral over D if there is $c \in D\backslash(0)$ such that $cy^m \in D$ for all even (odd) natural numbers m. For a somewhat less obvious corollary let us call an infinite subset S of N covered by an arithmetic sequence $T = \{t_n = \alpha + (n-1)\delta\}_{n=1}^{\infty}$, where α, δ are positive integers, if for each $n \in N$ we can find $s \in S$ such that $t_n \leq s \leq t_{n+1}$.

Corollary 14. If for $y \in K$ there is a $c \in D\backslash(0)$ such that $cy^s \in D$ for all s in a set S of natural numbers such that S is covered by an arithmetic sequence then y is almost integral over D.

Proof. Set $y = \frac{a}{b}$, let $T = \{t_n = \alpha + (n-1)\delta\}_{n=1}^{\infty}$, where α, δ are positive integers, be the sequence that covers S and set $\mu = \max(\alpha, 2\delta)$.Then $c_1 = b^\mu c$ has the property that $c_1 y^n \in D$ for all natural numbers n. (Note that because T covers S and because S can be put in a sequence, the distance between any consecutive members of S is less than or equal to 2δ).