QUESTION (**HD1102**) When, in a commutative ring R, is a prime ideal P an intersection of finitely many distinct prime ideals?

ANSWER: I assume that when you say "in a commutative ring R" you mean "in a commutative ring with $1 \neq 0$". With that assumption in place the answer is: Precisely when P is one of the intersecting primes and properly contained in the rest. To see this let $P = P_1 \cap P_2 \cap \ldots \cap P_n$ where $P, P_1, P_2, \ldots, P_n$ are prime ideals of the ring R. Then as $P_1 P_2 \ldots P_n \subseteq P = P_1 \cap P_2 \cap \ldots \cap P_n = P$ and as P is a prime ideal, one of the P_i, say P_j, is contained in P. Since already $P \subseteq P_j$ we conclude that $P = P_j$. That P is properly contained in the rest follows from the fact that P_i are all distinct. Indeed it is clear that if any of $P_1, P_2, \ldots, P_n$ say P_j is properly contained in the rest then $P_1 \cap P_2 \cap \ldots \cap P_n = P_j$.

What is interesting about your question is that the dual of the answer is also true and quite useful. Let me state the dual as the following proposition.

Proposition A. Let R be a commutative ring with 1. Given that $P, P_1, P_2, \ldots, P_n$ are distinct prime ideals of R. Then $P = P_1 \cup P_2 \cup \ldots \cup P_n$ if and only if $P = P_j$ for some $j \in \{1, 2, \ldots, n\}$.

Proof. $P = P_1 \cup P_2 \cup \ldots \cup P_n$ implies that $P \subseteq P_1 \cup P_2 \cup \ldots \cup P_n$ and by the well known prime avoidance lemma P must be contained in one of P_i, say P_j. But as $P = P_1 \cup P_2 \cup \ldots \cup P_n$, $P \supseteq P_j$. So $P = P_j$. The rest and the converse are easy.

The prime avoidance lemma is quite a celebrated result and can be found in any standard book but for your convenience a form of this lemma will be included at the end. Now the application of Proposition A, which may not be found in standard textbooks.

Corollary B. Let R be a commutative ring with 1 such that R has only finitely many prime ideals. Then every prime ideal of R is a minimal prime ideal of a principal ideal of R.

Proof. By Theorem 10 (page 6) of Kaplansky [K], for every principal ideal xR in a prime ideal P, there is a prime ideal $P(x)$ contained in P such that $P(x)$ is minimal over xR. Now let M be a prime ideal of R. Then $M = \bigcup_{x \in M} M(x)$ where $M(x)$ denotes the minimal prime of xR in M. But since there are only finitely many prime ideals in R, there are only finitely many x_i $(i = 1, \ldots, r)$ such that $M(x_i)$ are all distinct. So, $M = \bigcup_{i=1}^{n} M(x_i)$ and by Proposition A, $M = M(x_i)$ for some i, i.e., M is a minimal prime of a principal ideal.

The above corollary can be extended to do some more via the following somewhat contrived result. But let us note that commonly $Spec(R)$ is used to denote the set of all prime ideals of a ring R.

Proposition C. Let R be a commutative ring with 1. Let $\Delta \subseteq Spec(R)$ and suppose that for each nonunit $x \in R$ there is a $P \in \Delta$ such that $x \in P$. If $|\Delta| < \infty$ then every maximal ideal of R is in Δ and if all members of Δ are mutually incomparable then Δ consists precisely of the maximal ideals of R.

Proof. Let $\Delta = \{P_1, P_2, \ldots, P_n\}$. By the condition all nonunits of R are contained in $P_1 \cup P_2 \cup \ldots \cup P_n$. So for any maximal ideal M we have $M \subseteq P_1 \cup P_2 \cup \ldots \cup P_n$ and by prime avoidance $M \subseteq P_j$ for some j. But since M is a maximal ideal and P_j a proper ideal we conclude that $M = P_j$. Now let, by a rearrangement, $P_i = M_i$ for $i = 1, \ldots r \leq n$, where M_i are all the maximal ideals. If $r < n$ then obviously $P_{r+1} \subseteq M_i$ for some $i = 1,\ldots r$ because every ideal must be contained in a maximal ideal.

There are more applications of Proposition C if you are familiar with (or are prepared to read about) the so called star operations. (For a quick review of star operations look up sections 32 and 34 of Gilmer's book [G].) For our purposes here is a brief review.

Let $F(D)$ denote the set of all nonzero fractional ideals of an integral domains D. A star operation $*$ on an integral domain D is a function $*\colon F(D) \to F(D)$ such that for all $A,B \in F(D)$ and for all $0 \neq x \in K$

(a$_*$) $(x)^* = (x)$ and $(xA)^* = xA^*$,

(b$_*$) $A \subseteq A^*$ and $A^* \subseteq B^*$ whenever $A \subseteq B$,

(c$_*$) $(A^*)^* = A^*$.

A fractional ideal $A \in F(D)$ is called a $*$-*ideal* if $A = A^*$ and a $*$-ideal of finite type if $A = B^*$ where B is a finitely generated fractional ideal. Clearly a principal fractional ideal is a $*$-ideal for every star operation $*$ by (a$_*$). A star operation $*$ is said to be of finite character if $A^* = \bigcup\{B^* \mid 0 \neq B$ is a finitely generated fractional subideal of $A\}$. To ensure that, for $*$ of finite character, $0 \neq A$ is a star ideal it is enough to check that for each nonzero finitely generated ideal $I \subseteq A$ we have $I^* \subseteq A$. To each star operation $*$ we can associate an operation $*_s$ defined by $A^{*_s} = \bigcup\{B^* \mid 0 \neq B$ is a finitely generated subideal of $A\}$, for all $A \in F(D)$. It is easy to see that for a finitely generated $A \in F(D)$ we have $A^{*_s} = A^*$.

For $A \in F(D)$ define $A^{-1} = \{x \in K \mid xA \subseteq D\}$. It is easy to see that $A^{-1} \in F(D)$. The most well known examples of star operations are: the v-operation defined by $A \mapsto A_v = (A^{-1})^{-1}$, the t-operation defined by $A \mapsto A_t = \bigcup\{B_v \mid 0 \neq B$ is a finitely generated subideal of $A\}$, and the d-operation defined by $d : A \mapsto A$. Given two star operations α,β we say that α is finer than β, and denote it by $\alpha \leq \beta$ if for all $A \in F(D)$ we have $A^\alpha \subseteq A^\beta$. For further details you may look up sections 32 and 34 of Gilmer's book [G]. I will mention facts assuming that you are familiar with them.

By its definition t is of finite character (in fact $t = v_s$), $t \leq v$ while $\rho \leq t$ for every star operation ρ of finite character. If $*$ is a star operation of finite character, then using Zorn's Lemma we can show that a proper integral ideal maximal w.r.t. being a star ideal is a prime ideal and that every proper integral $*$-ideal is contained in a maximal $*$-ideal. Let us denote the set of all maximal t-ideals by $t-max(D)$. It can also be easily established that
$$D = \bigcap_{M \in t-\max(D)} D_M.$$

Corollary D. [M, Proposition 2.2] Let D be an integral domain and let $*$ be a finite character star operation on D. If D has only a finite number of distinct maximal $*$-ideals $M_1,M_2,\ldots M_n$ then $M_1,M_2,\ldots,M_n$ are precisely the maximal ideals.

For the proof all you need is the fact that every proper principal ideal xD, of D, being a $*$-ideal is contained in one of the maximal $*$-ideals M_i. Now for $* = d$ which is a finite character operation, nothing is gained except that the domain D has only finitely many maximal ideals. Yet if $* = t$. We get the conclusion that every maximal ideal of D is a t-ideal, and that D is semiquasi-local. Some applications of this observations can be seen in a paper by Mimouni [M].

An integral domain D is said to be a GCD domain if every pair $a,b \in D\backslash\{0\}$ has

$GCD(a,b) \in D.$ It can be easily established that in a GCD domain D, $GCD(x_1,x_2,\ldots,x_n)D = (x_1,x_2,\ldots,x_n)_v$. According to Sheldon [S], a nonzero prime ideal P of a GCD domain D is a PF-prime if for all $x,y \in P\backslash\{0\}$, $GCD(x,y) \in P$. Using the definition of t-ideals it is easy to see that in a GCD domain a prime ideal is a t-ideal if and only if it is a PF prime. It is easy to see that in a GCD domain D a nonzero prime ideal P is a PF-prime if and only if D_P is a valuation domain. An integral domain whose finitely generated ideals are principal is called a Bezout domain. Clearly every nonzero finitely generated ideal of a Bezout domain being principal is a v-ideal and hence every nonzero ideal of a Bezout domain is actually a t-ideal. It is well known that if D is a Bezout domain then for each nonzero prime ideal P, D_P is a valuation domain. Thus every maximal ideal of a Bezout domain is a PF-prime. With this introduction we prove the following corollary.

Corollary E. A GCD domain D is a semi-quasi-local Bezout domain if and only if D has finitely many maximal PF-primes.

Proof. If D is a semi-quasi-local Bezout domain then as every maximal ideal of D is a PF-prime and so there are only finitely many maximal PF-primes. If on the other hand D is a GCD domain with only finitely many maximal PF-primes say $P_1,P_2,\ldots,P_n$ then since PF-primes are prime t-ideals, by Proposition C, $P_1,P_2,\ldots,P_n$ are precisely the maximal ideals. But then $D = \bigcap_{i=1}^{n} D_{P_i}$ which forces D to be a Bezout domain, by [K, Theorem 107], since each of D_{P_i} is a valuation domain.

Call a prime ideal P of a domain D essential if D_P is a valuation domain. Now we see that thanks to Theorem 107 of Kaplansky [K], we can get more.

Corollary F. Let D be such that D has only finitely many maximal t-ideals and D_P is a valuation domain for every maximal t-ideal P of $D.$ Then D is a semi-quasi-local Bezout domain.

Now an integral domain D such that D_P is a valuation domain for every maximal t-ideal P is called a Prufer v-multiplication domain (PvMD). Alternatively a domain D is a PvMD if for every nonzero finitely generated ideal A, of D, A is t-invertible, i.e., we have $(AA^{-1})_t = D.$ A PvMD generalizes a Prufer domain (every nonzero finitely generated ideal is invertible) and it is well known that D is a Prufer domain if and only if D_P is a valuation domain for every maximal ideal. Indeed a GCD domain is a PvMD and a Bezout domains is Prufer. An integral domain D is called a generalized GCD domain if $aD \cap bD$ is invertible for each pair $a,b \in D\backslash\{0\}.$ The GGCD domains were introduced by Dan and David Anderson in [AA]. It can be shown that a D is GGCD if and only if D is a locally GCD PvMD. Looking at Corollaries E and F, and the above information it is easy to prove the following result.

Corollary G. A PvMD (GCD, GGCD) D is a semi-quasi-local Bezout domain if and only if D has finitely many maximal t-ideals.

Now the GCD domains have been generalized to almost GCD (AGCD) domains as follows. An integral domain D is an AGCD domain if for each pair $a,b \in D\backslash\{0\}$ there is a natural number n such that $a^nD \cap b^nD$ is principal, [Z]. Equivalently, D is an AGCD domain if for each pair $a,b \in D\backslash\{0\}$ there is a natural number n such that $(a^n,b^n)_v$ is principal. In [AZ] an integral domain D was called almost Bezout (Prufer) if for each pair $a,b \in D\backslash\{0\}$ there is a natural number n such that (a^n,b^n) is principal (repectively, invertible). It was shown in

[AZ] that D is almost Prufer if and only if for each maximal ideal M, D_M is an almost valuation domain (for each pair $a, b \in D\backslash\{0\}$ there is a natural number n such that $a^n \mid b^n$ or $b^n \mid a^n$).

Following [AZ] let's call a domain D an AGGCD domain if for each pair $a, b \in D\backslash\{0\}$ there is a natural number n such that $a^nD \cap b^nD$ is invertible. Rebecca Lewin [Le] gave AGGCD domains a detailed treatment. Recently, Li Qing [Li] has introduced the notion of an almost PvMD as a domain D such that for each pair $a, b \in D\backslash\{0\}$ there is a natural number n such that (a^n, b^n) is t-invertible. Equivalently, D is an almost PvMD (APvMD) if for each pair $a, b \in D\backslash\{0\}$ there is a natural number n such that $a^nD \cap b^nD$ is t-invertible. Indeed as invertible is t-invertible and as nonzero principal is invertible, AGCD and AGGCD domains are all APvMDs. She has also shown that [Li Theorem 2.3] D is an APvMD if and only if D_M is an almost valuation domain for every maximal t-ideal M. With this information we can now state the following corollary.

Corollary H. Given that D is an APvMD (AGCD, AGGCD) D is a semi-quasi-local almost Bezout domain if and only if D has finitely many maximal t-ideals.

Proof. It is sufficient to prove the result for APvMDs. By Proposition C the APvMD has finitely many maximal ideals each of which is a t-ideal. This leads to D being an intersection of a finite number of almost valuation domains with the same quotient field, which makes D into an almost Bezout domain by Theorem 3 of [AZ2]. The converse is obvious.

The prime avoidance Lemma

Kaplansky [K, page 55] attributes the prime avoidance lemma to N.H. McCoy and uses his terminology. On the other hand Eisenbud [E] states it in a way that is more in line with current terminology.

Prime avoidance Lemma ([E, Lemma 3.3]: Let R be a commutative ring with 1. Suppose that $I_1, I_2, \ldots, I_n, J$ are ideals of R and suppose that $J \subseteq I_1 \cup I_2 \cup, \ldots, \cup I_n$. Then J is contained in at least one of the I_i provided (a) R contains an infinite field or (b) at most two of the ideals I_j are not primes.

Proof. Part (a) is direct because no subspace of a vector space over an infinite field can be expressed as a union of finitely many proper subspaces. For part (b) we look at cases $n = 1, 2$ and $n > 2$. The case of $n = 1$ needs no proof. For $n = 2$, consider $J \subseteq I_1 \cup I_2$. We can assume I_1, I_2 incomparable, or the case reduces to that of $n = 1$. Suppose that $J \nsubseteq I_1, I_2$. Then there exist $x_2 \in J \backslash I_1$ and $x_1 \in J \backslash I_2$. But as $J \subseteq I_1 \cup I_2$ we have $x_1, x_2 \in I_1 \cup I_2$. This forces $x_1 \in I_1$ and $x_2 \in I_2$. But then $x_1 + x_2 \in J$ such that $x_1 + x_2 \notin I_1$ because of x_2 and $x_1 + x_2 \notin I_2$ because of x_1. So $I \nsubseteq I_1 \cup I_2$ a contradiction. Now let $n > 2$ and suppose that the lemma holds for all $r < n$. Since $n > 2$, at least one of I_i is a prime. By renumbering we can assume that I_1 is a prime ideal. Now assume that $J \nsubseteq I_i$ for any $i = 1, \ldots, n$. By the induction hypothesis then, J is not contained in any unions of proper subsets of $\{I_1, I_2, \ldots, I_n\}$. So, in particular, $J \nsubseteq \bigcup_{i \neq j} I_i$. This means there must be $x_j \in J \backslash \bigcup_{i \neq j} I_i$. But as $x_j \in J$ we must have $x_j \in I_j$ and so $x_1 \in I_1$ and not in $\bigcup_{i \neq 1} I_i$ and by similar arguments, each of $x_2, x_3, \ldots, x_n \notin I_1$, which is a prime, and so $x_2x_3\ldots x_n \notin I_1$. Now $x_2x_3\ldots x_n \in I_2, I_3, \ldots, I_n$ and with this information we conclude that $x_1 + x_2x_3\ldots x_n \notin I_i$ for $i = 1, 2, 3, \ldots, n$. Thus $J \not\subseteq I_1 \cup I_2 \cup, \ldots, \cup I_n$, a contradiction resulting from assuming that J is not contained in any of the $I_1, I_2, \ldots, I_n$.

I am thankful to Tiberiu Dumitrescu for giving this answer a critical read. From the kind of errors he caught it appears that the stroke that I had recently took a long time materializing. Perhaps my performance had been substandard for some time before I had the stroke.

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