ALMOST SPLITTING SETS AND AGCD DOMAINS

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Abstract. Let $D$ be an integral domain. A multiplicative set $S$ of $D$ is an almost splitting set if for each $0 \neq d \in D$, there exists an $n = n(d)$ with $d^n = st$ where $s \in S$ and $t$ is $v$-coprime to each element of $S$. An integral domain $D$ is an almost GCD (AGCD) domain if for every $x,y \in D$, there exists a positive integer $n = n(x,y)$ such that $x^n D \cap y^n D$ is a principal ideal. We prove that the polynomial ring $D[X]$ is an AGCD domain if and only if $D$ is an AGCD domain and $D[X] \subseteq D'[X]$ is a root extension, where $D'$ is the integral closure of $D$. We also show that $D + XD_S[X]$ is an AGCD domain if and only if $D$ and $D_S[X]$ are AGCD domains and $S$ is an almost splitting set.

1. Introduction

Let $D$ be an integral domain with quotient field $K$ and $D'$ the integral closure of $D$. By an overring of $D$ we mean a ring between $D$ and $K$. $D$ is said to be an almost GCD (AGCD) domain if for every $x,y \in D$, there exists $n = n(x,y) \in \mathbb{N}^*$ such that $x^n D \cap y^n D$ is a principal ideal. AGCD domains were introduced by the third author in [1] as a generalization of GCD domains (also see [2] and [3]). If $D$ is an AGCD domain, then $D'$ is an AGCD domain [1, Theorem 3.4] and $D \subseteq D'$ is a root extension (i.e., for each $x \in D'$ there exists a positive integer $n$ such that $x^n \in D$) [1, Theorem 3.1]. By [1, Theorem 5.6], an integrally closed domain $D$ is an AGCD domain if and only if the polynomial ring $D[X]$ is. The primary aim of this paper is to extend this result for arbitrary domains. We prove that $D[X]$ is an AGCD domain if and only if $D$ is an AGCD domain and $D[X] \subseteq D'[X]$ is a root extension.

Recall that for a nonzero fractional ideal $I$ of $D$, $I_v = (I^{-1})^{-1} = (D : I) : I = \bigcap \{xD \mid xD \supseteq I, x \in K\}$ and $I_v = \bigcup \{J_v \mid 0 \neq J \subseteq I \text{ is finitely generated}\}$. Hence, if $I$ is finitely generated, $I_v$ is finitely generated. It is well known that for $x,y \in D^* = D - \{0\}$, $x D \cap y D$ is a principal ideal if and only if $(x,y)_v$ is principal (indeed, $(x,y)_v$ is principal $\iff (x,y)^{-1} = x^{-1} D \cap y^{-1} D = \frac{1}{xy} (xD \cap yD)$ is principal). Call two nonzero elements $x,y \in D$ $v$-coprime if $(x,y)_v = D$, or equivalently, if $x D \cap y D = xy D$ (see Proposition 2.2 for several other equivalences).

A saturated multiplicative set $S$ of $D$ is called an almost splitting set if for each nonzero $x \in D$, there is a natural number $n = n(x)$ such that $x^n = ds$, where $s \in S$ and $d$ is $v$-coprime to every member of $S$. We also prove that for a saturated multiplicative set $S$, the composite polynomial ring $D + XD_S[X] = \{f(X) \in DS[X] \mid f(0) \in D\}$ is an AGCD domain if and only if $D$ and $D_S[X]$ are AGCD domains and $S$ is an almost splitting set.

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2. Almost Splitting Sets

In this section we investigate almost splitting sets. However, we begin by reviewing the notion of a splitting set. A saturated multiplicative set $S$ of $D$ is said to be a splitting set if for each $d \in D^*$ we can write $d = sa$ for some $s \in S$ and $a \in D$ with $s'D \cap aD = s'aD$ for all $s' \in S$, i.e., $s'$ and $a$ are $v$-coprime. The set $T = \{ t \in D^* \mid sD \cap tD = stD \text{ for all } s \in S \}$ is also a splitting set, called the $m$-complement of $S$. Each $d \in D^*$ has a unique representation (up to unit factors) $d = st$, where $s \in S$ and $t \in T$. If $d = st \ (s \in S, \ t \in T)$, then $dD_S \cap D = tD$. In fact, a saturated multiplicative set $S$ of $D$ is a splitting set if and only if $dD_S \cap D$ is principal for each $d \in D^*$. For these, and other, results on splitting sets, see [4]. Splitting sets are investigated further in [5].

Splitting sets can also be viewed in the context of the group of divisibility $G(D) = K^*/U(D)$ of $D$. Here $U(D)$ denotes the group of units of $D$ and $G(D)$ is partially ordered by $aU(D) \leq bU(D) \iff a|b$ in $D$. Note that $G(D)$ is order-isomorphic to $P(D)$ the multiplicative group of nonzero principal fractional ideals of $D$ ordered by inverse inclusion: $aU(D) \leftrightarrow aD$. Mott [6, Theorem 2.1] showed that there is a one-to-one correspondence between the set of convex directed subgroups of $P(D) \cong G(D)$ and the set of saturated multiplicative closed subgroups of $D$. The correspondence is given as follows. If $S$ is a saturated multiplicative closed subset of $D$, then $\langle S \rangle = \{ s_1 s_2^{-1}D \mid s_1, s_2 \in S \}$ is a convex directed subgroup of $P(D)$ with positive cone $\langle S \rangle_+ = \{ sD \mid s \in S \}$. In $G(D)$, we may identify $\langle S \rangle$ with $U(D_S)/U(D)$. In [7], Mott and Schenzkyd considered the question of when $\langle S \rangle \cong U(D_S)/U(D)$ is a cardinal summand of $P(D) \cong G(D)$, that is, when there is a subgroup $H$ of $P(D)$ with $\langle S \rangle \oplus_c H = P(D)$. In our terminology, they showed that $\langle S \rangle$ is a cardinal summand if and only if $S$ is a splitting set.

A splitting set $S$ is said to be an lcm splitting set if for each $s \in S$ and $d \in D$, $sD \cap dD$ is principal, or equivalently $D_T$, where $T$ is the $m$-complement of $S$, is a GCD-domain. Perhaps the most important example of an lcm splitting set is as follows. A set $\{p_\alpha\}$ of nonzero principal primes is a splitting set of principal primes if (a) for each $\alpha$, $\bigcap_{\alpha=1}^\infty p_\alpha^n D = 0$ (or equivalently, $\text{ht} \ p_\alpha D = 1$), and (b) for any sequence $\{p_\alpha\}$ of nonassociate members of $\{p_\alpha\}$, $\bigcap_{\alpha=1}^\infty p_\alpha D = 0$. Then $S = \{ up_\alpha \cdots p_{\alpha_n} \mid u \in U(D), \ p_\alpha, \in \{p_\alpha\}, \ n \geq 0 \}$ is an lcm splitting set [4, Proposition 2.6].

We next introduce the notion of an almost splitting set and an almost lcm splitting set.

**Definition 2.1.** Let $S$ be a saturated multiplicative set of an integral domain $D$. Then $S$ is an almost splitting set if for each $d \in D^*$, there is an $n = n(d)$ with $d^n \in D$ such that $d^nD$ is principal. An almost splitting set $S$ is an almost lcm splitting set if for all $s \in S$ and $d \in D^*$, there is an $n = n(s,d)$ such that $d^nD \cap s^nD$ is principal.

But first, we investigate the notions of $v$-coprimeness and $m$-complements more closely. The proof of the next proposition is straightforward and left to the reader. Proofs of several of the implications may be found in [4] and [5].

**Proposition 2.2.** Let $S$ be a (not necessarily saturated) multiplicative set of the domain $D$. Then for $t \in D^*$, the following are equivalent.

\begin{enumerate}
  \item $(s,t)_c = D$ for all $s \in S$,
  \item $sD \cap tD = stD$ for all $s \in S$,
\end{enumerate}
(3) \( t D_S \cap D = t D \),
(4) \( D_S \cap D_t = D \) where \( D_t = \{ d/t^n \mid d \in D, \ n \geq 0 \} \), and
(5) \( D_t : s = D t \) for all \( s \in S \).

**Definition 2.3.** For a nonempty subset \( S \) of an integral domain \( D \), let \( S^\perp = \{ x \in D^* \mid (x, s)_t = D \) for all \( s \in S \} \).

Note that by Proposition 2.2, \( D_S \cap D_{S^\perp} = D \). It is easily checked that for an integral domain \( D \) and \( \emptyset \neq S_1 \subseteq S_2 \subseteq D^* \), \( S_1^\perp \supseteq S_2^\perp \), and \( S_1 \subseteq S_1^{\perp\perp} \).

**Proposition 2.4.** Let \( S \) be a nonempty subset of nonzero elements of an integral domain \( D \). Then \( S^\perp \) is a saturated multiplicative set of \( D \) with \( \hat{S} \subseteq S^{\perp\perp} \) and \( S \cap S^{\perp\perp} = U(D) \) where \( \hat{S} \) is the saturation of the multiplicative set generated by \( S \). Moreover, \( S^\perp = S^{\perp\perp} \).

**Proof.** Let \( x, y \in S^\perp \) and \( s \in S \). Then \( (x, s)_v = (y, s)_v = D \), so \( D = ((x, s)(y, s))_v = (x y, x s, y s, s^2)_v \subseteq (x y, s)_v \subseteq D \) and hence \( (x y, s)_v = D \). Thus \( x y \in S^\perp \). Clearly, \( S^\perp \) is a saturated multiplicative set with \( S \subseteq S^{\perp\perp} \) and \( S \subseteq \hat{S} \subseteq S^{\perp\perp} \). Similarly, \( S \cap S^{\perp\perp} = U(D) \).

Note that \( S \subseteq S^{\perp\perp} \) gives \( S^\perp \supseteq S^{\perp\perp} \) and \( S^\perp \subseteq (S^{\perp\perp})^{\perp\perp} \), so \( S^\perp = S^{\perp\perp} \).

Suppose that \( S \) is an almost splitting set. Let \( x \in S^{\perp\perp} \), so there exists an \( n \geq 1 \) with \( x^n = st \) where \( s \in S \) and \( t \in S^\perp \). Then \( st = x^n \in S^{\perp\perp} \Rightarrow t \in S^{\perp\perp} \Rightarrow t \in S^{\perp\perp} \cap S^{\perp\perp} = U(D) \). So \( x^n \in S \) and hence \( x \in S \). Thus \( S = S^{\perp\perp} \).

However, in general we need not have \( S = S^{\perp\perp} \) even when \( S \) is a saturated multiplicative set generated by principal primes.

**Example 2.5.** Let \( (V, (p)) \) be a discrete valuation ring of rank greater than one. Then \( S = \{ u p^n \mid n \geq 0, u \in U(V) \} \) is a saturated multiplicative set of \( V \). Here \( S^\perp = U(D) \) and hence \( S \subseteq S^{\perp\perp} = D^* \).

The use of \( S^\perp \) may remind some readers of the set of orthogonal elements of a partial orderings of a set \( S \) in a partially ordered group or in a Riesz space (i.e., the set of all positive elements \( a \) with \( a \wedge s = 0 \) for each \( s \in S \)). Viewing an integral domain in the context of its group of divisibility, which is a partially ordered group, we see that the notion of \( v \)-coprimality is precisely the same as that of orthogonality.

We next give an example of an almost splitting set which generalizes the notion of a splitting set of primes.

**Example 2.6.** Let \( \{ P_\alpha \}_{\alpha \in \Lambda} \) be a nonempty collection of height-one prime ideals of an integral domain \( D \) with \( \bigcap_{n=1}^{\infty} P_{\alpha_n} = 0 \) for any countable subcollection. For each \( \alpha \in \Lambda \), assume that some \( (P_\alpha^n)_t \) is principal; say \( (P_\alpha^n)_t = (q_\alpha) \). Let \( S = \{ q_\alpha^{l_\alpha} \cdots q_\alpha^{l_n} \mid \alpha \in \Lambda, \ l_\alpha \geq 0 \} \) and let \( \hat{S} \) be the saturation of \( S \). Then for \( 0 \neq d \in D \), there exists an \( n \geq 1 \) with \( d^n = st \) where \( s \in S \) and \( t \in S^\perp \). Hence \( \hat{S} \) is an almost splitting set.

Since each \( P_\alpha \) is \( t \)-invertible, if \( I \) is a nonzero ideal contained in \( P_\alpha \), we get \( I_t = (P_\alpha J)_t \) with \( J = P_\alpha^{-1} I \). We repeatedly use this factorization property starting with \( I = dD \). By our height-one and intersection assumptions on the \( P_\alpha \)'s, we get \( dD = (P_\alpha \cdots P_{\alpha_n} J)_t \) for some \( \alpha_1, \ldots, \alpha_n, n \geq 0 \) and some ideal \( J \) contained in no \( P_\alpha \). As \( (P_\alpha^n)_t = q_\alpha^{dD} \) and \( q_\alpha \in S \), \( d^k D = s(J^k)_t \) for some \( k \geq 1 \) and \( s \in S \). So \( (J^k)_t = fD \) for some \( f \in D \). Then \( f \notin \bigcup P_\alpha \), hence \( (P_\alpha, f)_t = D \) for each \( \alpha \),
because $P_\alpha$ being $t$-invertible is a maximal $t$-ideal [8, Lemma 1]. As $(P_\alpha^n)_\iota = q_\alpha D$, we get $(q_\alpha, f)_\iota = D$ for each $\alpha$. Hence $f \in S^\perp$, because the $q_\alpha$’s generate $S$. Thus $d_k \in SS^\perp$. Hence $\tilde{S}$ is an almost splitting set. (Note: $\tilde{S} = \{us \mid u \in U(D), s \in S\}$ need not be saturated as is seen by taking a Dedekind domain $D$ with class group $Cl(D) = \mathbb{Z}_2$. Suppose that $M$ and $N$ are nonprincipal maximal ideals of $D$. Let $M^2 = (a), N^2 = (b)$ and $MN = (c)$, and let $S = \{a^nb^m \mid n, m \geq 0\}$. Then $(c^2) = M^2N^2 = (a)(b)$, so $c^2 = uab$ for some $u \in U(D)$. Thus $c^2 \in \tilde{S}$, but $c \notin \tilde{S}$.)

The following characterization of almost splitting sets similar to a characterization of splitting sets [4, Theorem 2.2] will be used.

**Proposition 2.7.** For a saturated multiplicative set $S$ of an integral domain $D$, the following are equivalent.

1. $S$ is an almost splitting set.
2. For $d \in D$, there exists an $n = n(d) \geq 1$ with $d^nDS \cap D$ principal.

**Proof.**

1. $\Rightarrow$ 2. Let $d^n = st$ where $s \in S$ and $t \in S^\perp$. Then $d^nDS \cap D = stDS \cap D = tdS \cap D = tD$ with the last equality following from Proposition 2.2.

2. $\Rightarrow$ 1. Let $0 \neq d \in D$. Suppose that $d^nDS \cap D = tD$. Then $tDS \cap D = d^nDS \cap D = tD$; so by Proposition 2.2, $t \in S^\perp$. Now $d^n = rt$ for some $r \in D$. Hence $rtDS = d^nDS = tD$, so $rDS = DS$. Since $S$ is saturated, $r \in S$. \qed

We next give a characterization of almost lcm splitting sets similar to the characterization of lcm splitting given in [4].

**Theorem 2.8.** For an almost splitting set $S$ of an integral domain $D$, the following conditions are equivalent.

1. $S$ is an almost lcm splitting set.
2. For $s_1, s_2 \in S$, there exists an $n = n(s_1, s_2) \geq 1$ with $s_1^nD \cap s_2^nD$ principal.
3. For $s_1, s_2 \in S$, there exists an $n = n(s_1, s_2) \geq 1$ and $s \in S$ with $s_1^nD \cap s_2^nD = sD$.
4. $DS_{\perp}$ is an AGCD-domain.

**Proof.**

1. $\Rightarrow$ 2. Clear. 2. $\Rightarrow$ 3. Let $s_1^nD \cap s_2^nD = xD$. Write $x^m = st$ where $s \in S$ and $t \in S^\perp$. Then $s_1^mD \cap s_2^mD = x^mD = stD$. Now $tD = tDS \cap D = stDS \cap D = xDS \cap D = (s_1^nD \cap s_2^nD)D \cap DS \cap D = DS \cap D = D$ implies $t \in U(D)$. So $s_1^mD \cap s_2^mD = sD$. (3) $\Rightarrow$ 4. Let $xDS_{\perp}, yDS_{\perp}$ be principal ideals of $DS_{\perp}$ where $x, y \in D^*$. Now $x^n = s_1t_1$ where $s_1 \in S$ and $t_1 \in S^\perp$ implies $x^nDS_{\perp} = s_1^nDS_{\perp}$ and likewise $y^nDS_{\perp} = s_2^nDS_{\perp}$ where $s_2 \in S$. Hence $x^mnDS_{\perp} = s_1^nD_{S_{\perp}}$ and $y^mn = s_2^nD_{S_{\perp}}$. Choose $l$ with $(s_1^m)^lD \cap (s_2^m)^lD = D$ where $s \in S$. Then $x^mnD_{S_{\perp}} \cap y^mnD_{S_{\perp}} = s_1^nD_{S_{\perp}} \cap s_2^nD_{S_{\perp}} = (s_1^nD \cap s_2^nD)D_{S_{\perp}} = sD_{S_{\perp}}$ is principal. So $D_{S_{\perp}}$ is an AGCD-domain. (4) $\Rightarrow$ 1. Let $s \in S$ and $d \in D^*$. So for some $n \geq 1$, $d^n = s_1t_1$ where $s_1 \in S$ and $t_1 \in S^\perp$. Choose $m \geq 1$ with $(s_1^m)^lD \cap (s_2^m)^lD = s^nD_{S_{\perp}} \cap s^nD_{S_{\perp}}$ principal. As in the proof of (2) $\Rightarrow$ (3), we can assume that $(s_1^m)^lD \cap (s_2^m)^lD = s^nD_{S_{\perp}}$ for some $s' \in S$. Then $s^md \cap d^mnD = s^md \cap s_1^mD = s^md \cap t_1^mD = (s^md_{S_{\perp}} \cap D) \cap t_1^mD = ((s^md_{S_{\perp}} \cap t_1^mD_{S_{\perp}}) \cap D) \cap t_1^mD = (s'\cap t_1^mD \cap D) \cap t_1^mD = s'^mD \cap t_1^mD = s'^mD \cap t_1^mD$ is principal. \qed

Recall that for an integral domain $D$, the $t$-class group of $D$ is $Cl_t(D) = T(D)/P(D)$ where $T(D)$ is the group of $t$-invertible $t$-ideals of $D$ and $P(D)$ is its subgroup of nonzero principal fractional ideals. When $S$ is a splitting set, there is a
natural isomorphism $\text{Cl}_t(D) \to \text{Cl}_t(D_S) \times \text{Cl}_t(D_{S^\perp})$ given by $[I] \to ([ID_S], [ID_{S^\perp}])$ where $[\ ]$ denotes the class of an ideal. See [4]. For $D$ a Krull domain, $\text{Cl}_t(D)$ is the usual divisor class group.

**Theorem 2.9.** Let $S$ be an almost splitting set in an integral domain $D$. Then the kernel of the canonical homomorphism $\theta: \text{Cl}_t(D) \to \text{Cl}_t(D_S) \times \text{Cl}_t(D_{S^\perp})$ given by $\theta([I]) = ([ID_S], [ID_{S^\perp}])$ is a torsion subgroup of $\text{Cl}_t(D)$.

**Proof.** It suffices to show that if $I$ is a nonzero integral ideal of $D$ with $ID_S$ and $ID_{S^\perp}$ principal, there is a $k \geq 1$ with $(I^k)_v$ principal. Suppose that $ID_S = a_1 D_S$ and $ID_{S^\perp} = a_2 D_{S^\perp}$ where $a_1, a_2 \in D$. Since $S$ is an almost splitting set, we can write $a_i^k = s_i t_i$ where $s_i \in S$ and $t_i \in S^\perp$, $i = 1, 2$. Then $I^k D_S = a_1^k D_S = s_1 t_1 D_S = t_1 D_S = s_2 t_2 D_{S^\perp}$. Likewise, $I^k D_{S^\perp} = s_2 t_2 D_S$. So $I^k D_S = a D_S$ and $I^k D_{S^\perp} = a D_{S^\perp}$ where $a = s_2 t_2 \in D$. But then $I^k \subseteq I^k D_S \cap I^k D_{S^\perp} = a D_S \cap a D_{S^\perp} = a (D_S \cap D_{S^\perp}) = aD$. So $I^k = J a$ for some ideal $J$ of $D$. Then $a D_S = I^k D_S = JD_S a D_S$, so $JD_S = D_S$. Likewise, $JD_{S^\perp} = D_{S^\perp}$. Let $x \in J^{-1}$; so $xJ \subseteq D$. Then $xJ D_S \subseteq D_S$ implies $x D_S \subseteq D_S$, so $x \in D_S$. Likewise, $x \in D_{S^\perp}$, so $x \in D_S \cap D_{S^\perp} = D$. Thus $J^{-1} \subseteq D$ and hence $J_v = D$. So $(I^k)_v = aD$ is principal.

Recall that a Krull domain $D$ is said to be almost factorial if $\text{Cl}_t(D)$ is torsion.

**Corollary 2.10.** Let $S$ be an almost lcm splitting set in an integral domain $D$. If $\text{Cl}_t(D_S)$ is torsion, then so is $\text{Cl}_t(D)$. If $D$ is root closed and $D_S$ is an AGCD domain, then $D$ is an AGCD domain. Hence if $D$ is a Krull domain with $D_S$ almost factorial, then $D$ is almost factorial.

**Proof.** By Theorem 2.8, $D_{S^\perp}$ is an AGCD domain and hence $\text{Cl}_t(D_{S^\perp})$ is torsion. Thus $\text{Cl}_t(D_S) \times \text{Cl}_t(D_{S^\perp})$ is torsion. By Theorem 2.9, ker $\theta$ is torsion. Then $\text{Cl}_t(D)$ itself is torsion.

Suppose that $D$ is root closed. Then $D_S$ is a root closed AGCD domain and hence is integrally closed. By the same reasoning, $D_{S^\perp}$ is integrally closed. Thus $D = D_S \cap D_{S^\perp}$ is integrally closed. By [1, Theorem 3.9], $D$ is an AGCD domain.

We end this section with a characterization of the integral domains with the property that every saturated multiplicative set is an almost splitting set. Recall that an integral domain $D$ is weakly Krull if $D = \bigcap_{ht P = 1} D_P$ where the intersection has finite character.

**Theorem 2.11.** An integral domain $D$ is weakly Krull with $\text{Cl}_t(D)$ torsion if and only if every saturated multiplicative set of $D$ is an almost splitting set.

**Proof.** ($\Rightarrow$) Suppose that every saturated multiplicative set of $D$ is an almost splitting set. By [9, Theorem 3.4], it suffices to show that if $P$ is a prime ideal minimal over a proper principal ideal $D$, then there is a natural number $n = n(x, P)$ with $x^n D_P \cap D$ principal. But since $S = D - P$ is an almost splitting set, this follows from Proposition 2.7.

($\Leftarrow$) Suppose that $D$ is a weakly Krull domain with $\text{Cl}_t(D)$ torsion and let $S$ be a saturated multiplicative set of $D$. Let $d$ be a nonzero nonunit of $D$. Since some power of $d$ is a product of primary elements [9, Theorem 3.4], it suffices to show that each nonzero primary element $q$ not in $S$ is in $S^\perp$. As $S$ is saturated, $q D$ is disjoint from $S$ and hence so is its radical. Since $qD$ is primary, $qD : s = qD$ for each $s \in S$. Thus $q \in S^\perp$.  \qed
3. Polynomial Extensions of AGCD Domains

Let $D$ be an integral domain and $S$ a multiplicative set of $D$. In this section we consider the question of when $D[X]$ or $D + XD_S[X]$ is an AGCD domain. We show (Theorem 3.4) that $D[X]$ is an AGCD domain if and only if $D$ is an AGCD domain and $D[X] \subseteq D'[X]$ is a root extension while (Theorem 3.12) $D + XD_S[X]$ is an AGCD domain if and only if $D$ and $D_S[X]$ are AGCD domains and $S$ is almost splitting.

Let $E$ be an overring of $D$. According to [2], we say that $D$ is $t$-linked under $E$, if whenever $x_1, \ldots, x_n \in D^*$ with $((x_1, \ldots, x_n)E)_v = E$, we have $((x_1, \ldots, x_n)D)_v = D$. We shall use the following result from [2].

**Lemma 3.1.** ([2, Theorem 5.9]) A domain $D$ is an AGCD domain if and only if

(i) $D'$ is an AGCD domain,

(ii) $D \subseteq D'$ is a root extension, and

(iii) $D$ is $t$-linked under $D'$.

**Remark 3.2.** By the proof of [2, Theorem 5.9], it follows that in Lemma 3.1 condition (iii) can be replaced by condition

(iii') whenever $x, y \in D^*$ are $v$-coprime in $D'$, $x, y$ are $v$-coprime in $D$.

We next prove that the $t$-linked-under property is stable under a polynomial base change.

**Proposition 3.3.** Let $D$ be a domain, $E$ an overring of $D$, and $K$ the quotient field of $D$. The following assertions are equivalent.

(a) $D$ is $t$-linked under $E$,

(b) $D[X]$ is $t$-linked under $E[X]$, and

(c) whenever $f, g \in D[X]^*$ are $v$-coprime in $E[X]$, then $f, g$ are $v$-coprime in $D[X]$.

**Proof.** (a) $\Rightarrow$ (c). For this implication, our argument is patterned after the proof of [10, Theorem 3.5]. Let $f, g$ be nonzero elements of $D[X]$ such that $f, g$ are $v$-coprime in $E[X]$. By [10, Theorem 3.2], $f, g$ are $v$-coprime in $E[X]$ if and only if $f, g$ are $v$-coprime in $K[X]$ and $(c(f)E + c(g)E)_v = E$, where $c(f)$ is the content ideal of $f$ in $D$. As $D$ is $t$-linked under $E$, $(c(f) + c(g))_v = D$. A new appeal to [10, Theorem 3.2] shows that $f, g$ are $v$-coprime in $D[X].$

(c) $\Rightarrow$ (b). Let $I$ be a nonzero finitely generated ideal of $D[X]$ such that $(IE[X])_v = E[X]$. By [11, Lemma 4.4], there exist $0 \neq a \in IE[X] \cap E$ and $f \in IE[X]$ with $c(f)_v = E$, where $c(f)$ is the content ideal of $f$; moreover, $((a, f)E[X])_v = E[X]$ for every such $a$ and $f$. Fix $a$ and $f$ as above. We can write $a = c_1h_1 + \cdots + c_mh_m$ with $c_1, \ldots, c_m \in E$ and $h_1, \ldots, h_m \in I$. As $E$ is an overring of $D$, there exists $0 \neq s \in D$ such that $sc_1, \ldots, sc_m \in D$. Then $0 \neq sa \in I \cap E = I \cap D$. Therefore, replacing $a$ by $sa$, we may assume that $0 \neq a \in I \cap D$. Since $f \in IE[X]$, we can write $f = b_0f_0 + \cdots + b_nf_n$ with $b_0, \ldots, b_n \in E$ and $f_0, \ldots, f_n \in I$. Choose $k$ greater than the degree of each $f_i$ and let $g = f_0 + f_1X^k + \cdots + f_nX^{nk} \in I$. Then $a, g$ are $v$-coprime in $E[X]$. Indeed, if this were not the case, the image of $g$ in $E[X]/\langle aE[X] \rangle = (E/aE)[X]$ is a zero divisor. Hence $dg \in \langle aE[X] \rangle$ for some $d \in E - aE$. Then $df_i \in aE[X]$ for each $i$, so $df_i \in aE[X]$, contradicting the fact that $a, f$ are $v$-coprime in $E[X]$. So $a, g \in I \subseteq D[X]$ are $v$-coprime in $E[X]$, hence $a, g$ are $v$-coprime in $D[X]$ by
our assumption. As $a, q \in I$, we get $I_v = D[X]$, thus (b) holds. The implication (b) $\Rightarrow$ (a) is an easy consequence of the definition.

The following theorem is the main result of this paper.

**Theorem 3.4.** Let $D$ be a domain. Then $D[X]$ is an AGCD domain if and only if $D$ is an AGCD domain and $D[X] \subseteq D'[X]$ is a root extension.

**Proof.** Assume that $D[X]$ is an AGCD domain. By Lemma 3.1, $D[X] \subseteq D'[X] = (D[X])'$ is a root extension. Now, let $a, b \in D^*$. As $D[X]$ is an AGCD domain, there exist a positive integer $n$ and $c \in D[X]$ such that $a^n D[X] \cap b^n D[X] = c D[X]$. Then $c \in D$ and it follows easily that $a^n D \cap b^n D = c D$. So $D$ is an AGCD domain.

Conversely, assume that $D$ is an AGCD domain and $D[X] \subseteq D'[X]$ is a root extension. By Lemma 3.1, $D$ is $t$-linked under $D'$, so $D[X]$ is $t$-linked under $D'[X]$ by Proposition 3.3. As $D$ is an AGCD domain, so are $D'$ and $D'[X]$, cf. [1, Theorems 3.4 and 5.6]. Hence $D[X]$ is an AGCD domain by Lemma 3.1.

**Example 3.5.** By [2, Theorem 4.17], $\mathbb{Z}[2i]$ is an AGCD domain with integral closure $\mathbb{Z}[i]$. An easy computation shows that $f^2 \in \mathbb{Z}[2i][X]$ for each $f \in \mathbb{Z}[i][X]$. By Theorem 3.4, $\mathbb{Z}[2i][X]$ is an AGCD domain.

However, as the following example shows, $D$ an AGCD domain need not always imply that $D[X]$ is an AGCD domain.

**Example 3.6.** If $m$ is a square-free integer $m \equiv 5 \pmod 8$ and $D = \mathbb{Z}[\sqrt{m}]$, then $D$ is an AGCD domain [2, Theorem 4.17], while $D[X]$ is not AGCD, because $D[X] \subseteq D'[X]$ is not a root extension. Indeed, if it were, then reducing modulo 2, $\mathbb{F}_2[X] \subseteq \mathbb{F}_2[X]$ would be a root extension, where $\mathbb{F}_2$ is the field with 2 elements (we have used the easy-to-obtain isomorphism $D'/2D' = \mathbb{F}_4$). But if $t \in \mathbb{F}_4 - \mathbb{F}_2$, it is easy to see that no power of $t + X$ lies in $\mathbb{F}_2[X]$.

This example can also serve to establish that if $D \subseteq E$ is a root extension, it is not necessary that $D[X] \subseteq E[X]$ should also be a root extension.

**Example 3.7.** Let $A$ be a GCD domain of characteristic 2. By the paragraph following [3, Remark 4.1], $D = A[Y^2, Y^3]$ is an AGCD domain and $D' = A[Y]$. As $D'[X] \subseteq D$, it follows that $(D'[X])^2 \subseteq D[X]$, so $D[X]$ is an AGCD domain by Theorem 3.4.

When $D$ contains a field, it is easy to describe when $D[X] \subseteq D'[X]$ is a root extension.

**Proposition 3.8.** Let $D \subseteq E$ be an extension of domains such that $D$ contains a field. Assume that $D[X] \subseteq E[X]$ is a root extension.

(a) If $D \supseteq \mathbb{Q}$, then $D = E$.

(b) If $D \supseteq \mathbb{F}_p$ with $p$ a positive prime, then $D \subseteq E$ is a purely inseparable extension.

**Proof.** Let $a \in E$. Then $(1 + aX)^n \in D[X]$ for some positive $n$. Hence $na, a^n \in D$. So (a) holds, because $n \in U(D)$ in this case.

Now assume that $D \supseteq \mathbb{F}_p$, and decompose $n$ as $n = p^rm$ with $(m, p) = 1$. As $(1 + aX)^n = (1 + a^{p^r}X^{p^r})^m$, we get $m a^{p^r} \in D$, so $a^{p^r} \in D$, because $m \in U(D)$. □
The above proposition provides two types of AGCD domains $D$ such that $D[X]$ an AGCD domain implies that $D[X_1, X_2, \ldots, X_n]$ is an AGCD domain for indeterminates $X_1, X_2, \ldots, X_n$. One is when $D$ contains the field of rational numbers. In this case, $D[X]$ being AGCD forces $D$ to be integrally closed and it is well known [1] that if $D$ is an integrally closed AGCD domain, then so is a polynomial ring over $D$ in several variables. The other case is when $D$ has positive characteristic. In this case, the $D[X]$ being an AGCD domain forces the integral closure of $D$ to be purely inseparable over $D$ and thus paves the way for $D[X_1, X_2, \ldots, X_n]$ to be AGCD. This leads us to the following question.

**Question 3.9.** Does $D[X]$ an AGCD domain imply that $D[X_1, X_2, \ldots, X_n]$ is an AGCD domain for finitely many indeterminates $X_1, X_2, \ldots, X_n$?

Apparently the answer to this question is related to the following question.

**Question 3.10.** If $D \subseteq E$ is a root extension such that $D[X] \subseteq E[X]$ is a root extension, is $D[X, Y] \subseteq E[X, Y]$ also a root extension?

Let $D$ be a domain and $S \subseteq D^*$ a multiplicative set. We consider the composite ring $D + XD_S[X] = \{f \in D_S[X] \mid f(0) \in D\}$. By [12, Corollary 1.5] or [13, Corollary 2.6] $D + XD_S[X]$ is a GCD domain if and only if $D$ is a GCD domain and $S$ is a splitting set. When $D$ is integrally closed, it was shown in [3, Theorem 3.1] that $D + XD_S[X]$ is an AGCD domain if and only if $D$ is an AGCD domain and $S$ is an almost splitting set. We extend this result to arbitrary domains. Nevertheless, in our proof we use the result cited above. We need the following lemma (see [14, Theorem 1] for a similar result).

**Lemma 3.11.** Let $D$ be an AGCD domain, $S \subseteq D^*$ a multiplicative set, and $T$ the saturation of $S$ in $D'$. If $S$ is almost splitting in $D$, then so is $T$ in $D'$.

**Proof.** Let $0 \neq a \in D'$. Since $D \subseteq D'$ is a root extension (Lemma 3.1) and $S$ is almost splitting in $D$, there exists a positive integer $n$ such that $a^n = bs$ with $b \in D$ $v$-coprime in $D$ to every element of $S$ and $s \in S$. By the paragraph before [2, Theorem 5.11], $b$ is also $v$-coprime in $D'$ to every element of $S$. Thus $b \in T^+$. Hence $T$ is an almost splitting set.

**Theorem 3.12.** Let $D$ be a domain and $S \subseteq D^*$ a saturated multiplicative set. Then $D + XD_S[X]$ is an AGCD domain if and only if

(i) $D$ and $D_S[X]$ are AGCD domains, and

(ii) $S$ is almost splitting.

**Proof.** Set $E = D + XD_S[X]$. Assume that $E$ is an AGCD domain. As in the proof of Theorem 3.4, we can show that $D$ is an AGCD domain. As shown in [1, Section 5], a ring of quotients of an AGCD domain is still an AGCD domain. Hence $ES = D_S[X]$ is an AGCD domain. So (i) holds. The fact that $S$ is almost splitting was shown in the proof of [3, Theorem 3.1]. For the convenience of the reader, we repeat the argument here. Let $0 \neq a \in D$. Since $E$ is an AGCD domain, there exists a positive integer $n$ and $s \in E$ such that $(a^n, X^n)_v = sE$. Since $s \mid a^n$, $s \in D$ and since $s \mid X^n$, $s \in S$ (because $S$ is saturated). Set $b = a^n/s \in D$ and let $t \in S$. Since $t \mid X^n/s$, we derive successively that $(b, X^n/s)_v = E$, $(b, t)_v = E$, and $(b, tD)_v = D$. Hence $a^n = bs$ with $b \in D$ $v$-coprime to every element of $S$ and $s \in S$, that is, $S$ is almost splitting.
Conversely, assume that (i) and (ii) hold. We prove that $E$ is an AGCD domain using Lemma 3.1 and Remark 3.2. Let $T$ be the saturation of $S$ in $D'$. By the preceding lemma, $T$ is almost splitting in $D'$. Since $D$ is an AGCD domain, $D'$ is an integrally closed AGCD domain [1, Theorem 3.4]. The integral closure of $E$ is $E' = D' + XD'_f[X] = D' + XD'_s[X]$, cf. [15, Theorem 2.7]. By [3, Theorem 3.1], $E'$ is an AGCD domain. For proving that $E \subseteq E'$ is a root extension, let $f \in E'$. As $D$ and $D_s[X]$ are AGCD domains, $D \subseteq D'$ and $D_s[X] \subseteq D'_s[X]$ are root extensions, cf. Lemma 3.1. So there exist two positive integers, $m$ and $n$, such that $f^m \in D_s[X]$ and $f(0)^mn \in D$. It follows that $f^mn \in E$. Thus $E \subseteq E'$ is a root extension. To complete the proof, it suffices to verify that condition (iii') of Remark 3.2 holds in $E$. Let $f, g \in E'$ such that $f, g$ are $v$-coprime in $E'$. As noted in [16], $E$ is the directed union (limit) of its subrings $D[X/s]$ for $s \in S$. Similarly, $E'$ is the directed union of $D'[X/s]$ for $s \in S$. Note that $D[X/s]$ is $D$-isomorphic to $D[X]$. Let $s \in S$ such that $f, g \in D[X/s]$. We claim that $f, g$ are $v$-coprime in $D[X/s]$. Indeed, $D'[X/s]$ is an AGCD domain [1, Theorem 5.6], so there exist a positive integer $k$ and $h \in D'[X/s]$ such that $((f^k, g^k)) = hD'[X/s]$. In particular, $h$ is a common divisor of $f$ and $g$ not only in $D'[X/s]$ but also in $E'$. As $f, g$ are $v$-coprime in $E'$, so are $f^k, g^k$. Then $h \in U(E') = U(D'[X/s]) = U(D')$. So $f^k, g^k$ are $v$-coprime in $D'[X/s]$. By part (5) of [1, Lemma 1.1], $f, g$ are $v$-coprime in $D'[X/s]$. By Proposition 3.3, $D[X/s]$ is $t$-linked under $D'[X/s]$, so $f, g$ are $v$-coprime in $D[X/s]$. Now since $f, g$ are $v$-coprime in every $D[X/s]$ that contains them, a direct limit argument shows that $f, g$ are $v$-coprime in $E$. \[\Box\]

**Corollary 3.13.** Let $D$ be a domain with quotient field $K$. Then $D + XK[X]$ is an AGCD domain if and only if $D$ is.

**References**


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