UMV-domains

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ABSTRACT: We call a domain R a *UMV-domain* if each nonzero prime ideal in the polynomial ring $R[X]$ satisfying $P \cap R = 0$ is a maximal divisorial ideal of $R[X]$. We characterize v-domains as integrally closed UMV-domains, and we provide characterizations of UMV-domains similar to those known for UMT-domains.

Introduction

Let R be an integral domain with quotient field K. If $f \in K[X]$ is an irreducible polynomial, then we call the prime ideal $P = fK[X] \cap R[X]$ an *upper to zero.* Uppers to zero have been used by many authors to characterize ring-theoretic properties. For example, it follows from [6, Theorem 19.15] that a domain R is a Prüfer domain if and only if $P \nsubseteq MR[X]$ for each upper to zero P in $R[X]$ and each maximal ideal M of R. A corresponding result exists for Prüfer v-multiplication domains (PVMDs): A domain R is a PVMD if and only if R is integrally closed and each upper to zero in $R[X]$ is a maximal t-ideal of $R[X]$ [10, Proposition 2.6]. This led the authors of the present article to isolate this condition on uppers to zero by defining *UMT-domains* to be those domains in which each upper to zero is a maximal t-ideal [12]. UMT-domains were further studied in [3] and [4]. The purpose of this article is to study *UMV-domains*, domains with the property that each upper to zero is a maximal v-ideal, that is, a maximal divisorial ideal.

In Section 1 we study uppers to zero. We then apply these results in Section 2 to our study of UMV-domains. We show that a domain R is a UMV-domain if and only if each upper to zero P satisfies $P^{-1} \neq R[X]$ and $c(P)_v = R$ (where $c(P)$ denotes the ideal of R generated by the coefficients of the polynomials in P). We characterize v-domains as integrally closed UMV-domains. We also point out where the similarities with UMT-domains fail to hold; to some extent the problem is that, while maximal t-ideals are plentiful (recall that for any domain R we have $R = \bigcap\{R_M \mid M \text{ is a maximal } t\text{-ideal of } R\}$), a domain may have no maximal divisorial ideals. In a brief third section, we pose several questions related to UMV-domains.

1 Uppers to zero

We begin by reviewing terminology. Let R be a domain with quotient field K. For fractional ideals I, J of R, $(I : J) = \{x \in K \mid xJ \subseteq I\}$, and $(I :_R J) = \{x \in R \mid xJ \subseteq I\}$. We write I^{-1} for $(R : I)$, and I_v for $(I^{-1})^{-1}$. Finally, $I_t = \bigcup A_v$, where the union is taken over all finitely generated subideals A of I. Most of the facts about the v- and t-operations which we shall use can be found in [6, Section 32].

Definition 1.1. Let $I = fK[X] \cap R[X]$, where f is a nonconstant polynomial in $K[X]$. Then I is said to be *almost principal* if there is a nonzero element $a \in R$ such that $aI \subseteq fR[X]$.

Lemma 1.2. *Let $I = fK[X] \cap R[X]$, where f is a nonconstant polynomial in $K[X]$. Then $I^{-1} \cap K[X] = (I : I)$.*

Proof. [7, Lemma 1.14]. □

Proposition 1.3. *Let $P = fK[X] \cap R[X]$ with f irreducible in $K[X]$. Then the following statements are equivalent.*

(1) P is almost principal.

(2) $P^{-1} \not\subseteq K[X]$.

(3) $P^{-1} \neq (P : P)$.

(4) There is an element $g \in R[X] \setminus P$ with $gP \subseteq fR[X]$.

(5) $P = (f :_{R[X]} a)$ for some $a \in R$.

(6) $P = (R[X] :_{R[X]} \psi)$ for some $\psi \in K(X)$.

Proof. The equivalence of the first four conditions is established in [7, Proposition 1.15]. Assume (1). Then we have $aP \subseteq fR[X]$ for some $0 \neq a \in R$. Thus $P \subseteq (f :_{R[X]} a)$. On the other hand, if $h \in R[X]$ satisfies $ha \in fR[X]$, then $ha \in P \subseteq fK[X]$; hence $h \in fK[X] \cap R[X] = P$. Hence (1) $\Rightarrow$ (5). (5) $\Rightarrow$ (6) follows from the fact that $(f :_{R[X]} a) = (R[X] :_{R[X]} af^{-1})$. Finally, assume (6). If $\psi \in K[X]$, then there is some element $a \in R$ with $a\psi \in R[X]$, whence $a \in P$, contradicting that $P \cap R = 0$. Thus $\psi = g/f$ for some $g \in R[X] \setminus fK[X]$. It follows that $g \notin P$, and we have $gP \subseteq fR[X]$, proving (4). □

The following result is the key to our characterization of UMV-domains in Theorem 2.2 below.

Lemma 1.4. *Let I be an ideal of $R[X]$, and assume that $I^{-1} \neq R[X]$ and that $I \cap R \neq 0$. Then $c(I)_v \neq R$.*

Proof. We first observe that $I^{-1} \subseteq K[X]$, since for any nonzero element $a \in I \cap R$, we have $aI^{-1} \subseteq R[X] \subseteq K[X]$. Now let $\psi \in I^{-1} \setminus R[X]$, and assume that ψ has minimal degree among elements of $I^{-1} \setminus R[X]$. We shall show that, in fact, ψ has degree zero. We shall then have $I \subseteq (R[X] :_{R[X]} \psi) = (R :_R \psi)R[X]$. It will then follow that $c(I) \subseteq (R :_R \psi)$; since $(R :_R \psi)$ is divisorial, this will imply that $c(I)_v \neq R$.

To see that ψ must have degree zero, write $\psi = u_n X^n + \cdots + u_0$. We claim that $(R :_R u_n) \subseteq (R :_R u_0) \cap \cdots \cap (R :_R u_{n-1})$. If not, pick $b \in R$ with $bu_n \in R$ but $bu_i \notin R$ for some $i < n$. Then $\psi' = b\psi - bu_n X^n$ is an element of $I^{-1} \setminus R[X]$ of degree smaller than that of ψ, a contradiction. Thus the claim is true. In particular, $u_n \notin R$. Now let $g \in (R[X] :_{R[X]} \psi)$, $g = a_m X^m + \cdots + a_0$. Then $a_m u_n \in R$, whence by the claim, $a_m u_i \in R$ for each i. That is, $a_m \in (R[X] :_{R[X]} \psi)$. It then follows that $g - a_m X^m \in (R[X] :_{R[X]} \psi)$. By induction on the degree of g, we obtain $a_i \in (R[X] :_{R[X]} \psi)$ for each i. In particular, $a_i u_n \in R$ for each i, i.e., $g \in (R :_R u_n)R[X]$. Hence $I \subseteq (R[X] :_{R[X]} \psi) \subseteq (R :_R u_n)R[X]$, and $u_n \in I^{-1} \setminus R[X]$. Therefore, $\psi = u_n$ has degree zero, as desired. □

Proposition 1.5. *Let R be a domain with quotient field K, and let $P = fK[X] \cap R[X]$ with f irreducible in $K[X]$. Consider the following conditions on P.*

(1) P is maximal divisorial.

(2) P is v-invertible.

(3) $(P : P) = R[X]$.

(4) $P^{-1} \cap K = R$.

(5) $c(P)_v = R$.

(6) P is almost principal.

(7) P is divisorial.

Then (1) $\Rightarrow$ (2) $\Leftrightarrow$ (3) $\Rightarrow$ (4) $\Leftrightarrow$ (5), and (1) $\Rightarrow$ (6) $\Rightarrow$ (7).

Proof. The implication (1) $\Rightarrow$ (2) is proved in [5, Proposition 2.5]. That (2) $\Rightarrow$ (3) is well known and is true more generally. Indeed, let I be a v-invertible ideal of a domain R. Then $(I : I) \subseteq (II^{-1} : II^{-1}) \subseteq ((II^{-1})_v : (II^{-1})_v) = R$ (by v-invertibility of I). To show that (3) $\Rightarrow$ (4), let $u \in P^{-1} \cap K$. By Lemma 1.2, $u \in (P : P) = R[X]$. Hence $u \in R[X] \cap K = R$, as desired. The equivalence (4) $\Leftrightarrow$ (5) is straightforward. Note that (6) $\Rightarrow$ (7) follows from Propositon 1.3.

(3) $\Rightarrow$ (2): We may assume that $P^{-1} \neq R[X]$. If P is not divisorial, then $P_v \cap R \neq 0$ (since $P_v \supsetneqq P$), and $(P_v)^{-1} = P^{-1} \neq R[X]$. By Lemma 1.4, $c(P)_v \subseteq c(P_v)_v \subsetneqq R$. This contradicts (5). Hence P is divisorial. Now let $\psi \in (PP^{-1})^{-1}$, so that $\psi PP^{-1} \subseteq R[X]$. Then $\psi P \subseteq P_v = P$, and $\psi \in (P : P) = R[X]$. Thus $(PP^{-1})^{-1} = R[X]$, and P is v-invertible.

(1) $\Rightarrow$ (6): By [5, Proposition 2.1], $P = (R[X] :_{R[X]} \psi)$ for some $\psi \in K(X) \setminus R[X]$, and P is almost principal by Proposition 1.3. $\square$

We do not know whether (4) $\Rightarrow$ (3) or (7) $\Rightarrow$ (6) in Proposition 1.5. However, other than these two possibilities, no implications other than those given exist. That condition (6) does not imply (5) can be illustrated with any Noetherian domain R having a divisorial prime P of height ≥ 2. This follows from the facts that $PR[X]$ necessarily contains an upper to zero and that uppers to zero in Noetherian domains are automatically almost principal [13, Proposition 3.3]. That condition (2) does not imply (7) is illustrated by any domain R allowing an upper to zero P with $P^{-1} = R[X]$. We produce such an example at the end of this section.

It is useful to observe that much of Proposition 1.5 holds for "non-prime uppers to zero", that is without the assumption that f is irreducible.

Proposition 1.6. *Let R be a domain with quotient field K, and let $I = fK[X] \cap R[X]$, where f is a nonconstant polynomial in $K[X]$. Consider the following conditions on I.*

(2) I is v-invertible.

(3) $(I : I) = R[X]$.

(4) $I^{-1} \cap K = R$.

(5) $c(I)_v = R$.

(6) I is almost principal.

(7) I is divisorial.

Then (2) $\Leftrightarrow$ (3) $\Rightarrow$ (4) $\Leftrightarrow$ (5), and (6) $\Rightarrow$ (7).

Proof. The only implication requiring proof is (6) $\Rightarrow$ (7). Assume (6), and choose a nonzero element $a \in R$ with $aI \subseteq fR[X]$. Then $aI_v \subseteq fR[X] \subseteq fK[X]$, whence $I_v \subseteq fK[X]$. Hence $I_v \subseteq fK[X] \cap R[X] = I$. $\square$

Let P be an upper to zero which satisfies $(P : P) = R[X]$ and $P^{-1} \neq R[X]$. Observe that the proof of (3) $\Rightarrow$ (2) of Proposition 1.5 shows that P is divisorial. Also, since $(P : P) = R[X]$, P is v-invertible. It then follows from [5, Proposition 2.5] that P is maximal divisorial. That is, we have the following result.

Proposition 1.7. *With the notation of Proposition 1.5, assume that $P^{-1} \neq R[X]$. Then conditions (1), (2), and (3) are equivalent.*

Proposition 1.8. *Assume that R is integrally closed. Then*

(a) with the notation of Proposition 1.5, conditions (1)-(5) are equivalent.

(b) with the notation of Proposition 1.6, conditions (2)-(5) are equivalent (and condition (6) actually holds).

Proof. (a) It suffices to prove (5) $\Rightarrow$ (1). Suppose that B is a divisorial ideal of $R[X]$ which properly contains P. Then $B \cap R \neq 0$. By [17, Lemme 2], $B = (B \cap R)R[X]$. Hence $c(B) = B \cap R$. Now $R = c(P)_v \subseteq c(B)_v = B \cap R$. It follows that $B = R$. Therefore, P is a maximal divisorial ideal.

(b) It suffices to prove (5) $\Rightarrow$ (2). Write $I = fc(f)^{-1}R[X]$ by [17, Lemme 1]. Then $c(I) = c(f)c(f)^{-1}$. Since $c(I)_v = R$, this shows that $c(f)$ is v-invertible. It follows that I is v-invertible. Also, $aI \subseteq fR[X]$ for any $a \in c(f)$, so (6) holds. □

We conclude this section with the example promised above. Although we could base the proof on Lemma 1.4, we instead use the following lemma, since it may be of some independent interest.

Proposition 1.9. *Let $P = fK[X] \cap R[X]$ be an upper to zero, and let I denote the ideal generated by the constant terms of the elements of P. If $I^{-1} = R$ and $P^{-1} \neq R[X]$, then P is almost principal.*

Proof. Suppose that P is not almost principal. Let $\psi \in P^{-1}$. Assuming that $I^{-1} = R$, we shall show that $\psi \in R[X]$. Since P is not almost principal, we have $\psi \in K[X]$. Let c denote the constant term of ψ. For $g \in P$, since $\psi g \in R[X]$, we have $ca \in R$, where a is the constant term of g. Hence $c \in I^{-1} = R$. In particular, if $\psi \in K$, then $\psi \in R$. Now, since $c \in R$, the polynomial $\omega = (\psi - c)/X) \in P^{-1}$. By induction, we have that $\omega \in R[X]$. But then $\psi \in R[X]$, as desired. □

Example 1.10. An example of an upper to zero with $P^{-1} = R[X]$. Thus P is trivially v-invertible but is not divisorial. Let k be a field, and let s, t be indeterminates over k. Set $R = k[\{st^{2^n} \mid n \geq 0\}]$. This example was discussed in [7] and [11]. In particular, [11] contains a direct proof that the upper to zero $P = (X-t)K[X] \cap R[X]$ (where K denotes the quotient field of R) is not almost principal. It is easy to see that a monomial $s^i t^j \in k[s,t]$ lies in R iff $i \geq \varphi(j)$, where $\varphi(j)$ is the number of 1's in the binary expansion of j. To see that $P^{-1} = R[X]$, it suffices to show that $I^{-1} = R$, where I is the ideal generated by the constant terms of P (Proposition 1.9). It is easy to see that $sX^{2^n} - st^{2^n} \in P$ for each $n \geq 1$. Thus $st^{2^n} \in I$ for each $n \geq 1$. Set $J = (\{st^{2^n} \mid n \geq 1\})$. Then $J \subseteq I$, and we need only show that $J^{-1} = R$. If $u \in J^{-1}$, then $ust \in R$, so that $u = a/st$ for some $a \in R$. We shall show that $a \in stR$. For this we may assume that a is a monomial, say $a = s^i t^j$. Since $a \in R$, we have $i \geq \varphi(j)$. Note that for each $n \geq 1$, we have $(a/st)st^{2^n} \in R$, whence $s^i t^{2^n-1+j} \in R$. It follows that $i \geq \varphi(2^n - 1 + j)$. For sufficiently large n, we have $\varphi(2^n - 1 + j) = 1 + \varphi(j-1)$, whence $i - 1 \geq \varphi(j-1)$. It follows that $a = (st)(s^{i-1}t^{j-1}) \in stR$, as desired.

2 UMV-domains

Definition 2.1. A domain R is a *UMV-domain* if each upper to zero in $R[X]$ is a maximal divisorial ideal of $R[X]$.

Theorem 2.2. *The following statements are equivalent for a domain R with quotient field K.*

(1) R is a UMV-domain.

(2) Each upper to zero P in $R[X]$ satisfies $(P : P) = R[X] \neq P^{-1}$.

(3) For each nonconstant polynomial $f \in K[X]$, the ideal $I = fK[X] \cap R[X]$ satisfies $(I : I) = R[X] \neq I^{-1}$.

(4) Each upper to zero P in $R[X]$ satisfies $P^{-1} \neq R[X]$ and $c(P)_v = R$.

(5) For each nonconstant polynomial $f \in K[X]$, the ideal $I = fK[X] \cap R[X]$ satisfies $I^{-1} \neq R[X]$ and $c(I)_v = R$.

Proof. (1) $\Rightarrow$ (2): Let R be a UMV-domain, and let P be an upper to zero. Then P is maximal divisorial, and we have $(P : P) = R[X]$ by Proposition 1.5. Since P is divisorial, we also have $P^{-1} \neq R[X]$.

(2) $\Rightarrow$ (3): Let f, I be as given, and (we may) assume $f \in R[X]$. Factor f in $K[X]$: $f = g_1^{m_1} \cdots g_r^{m_r}$, where each g_i is irreducible in $K[X]$, and set $P_i = g_i K[X] \cap R[X]$. Let $\psi \in (I : I)$; recall from Lemma 1.2 that this implies that $\psi \in K[X]$. Then

$$\psi(P_1^{m_1} \cdots P_r^{m_r}) \subseteq \psi I \subseteq I.$$

Thus

$$\psi(P_1^{m_1-1}P_2^{m_2}\cdots P_r^{m_r})P_1 \subseteq P_1.$$

Since $(P_1 : P_1) = R[X]$, this implies that

$$\psi(P_1^{m_1-1}P_2^{m_2}\cdots P_r^{m_r}) \subseteq R[X].$$

Rewrite this as

$$\psi(P_1^{m_1-2}P_2^{m_2}\cdots P_r^{m_r})P_1 \subseteq R[X].$$

Then, using (2) and Lemma 1.2, we have $\psi(P_1^{m_1-2}P_2^{m_2}\cdots P_r^{m_r}) \subseteq P_1^{-1} \cap K[X] = (P_1 : P_1) = R[X]$. Repeating this as many times as necessary, we eventually obtain $\psi \in R[X]$, as desired. Finally, $I^{-1} \supseteq P_1^{-1} \supsetneqq R[X]$.

(3) $\Rightarrow$ (5): Assume (3), and let $I = fK[X] \cap R[X]$ with $f \in R[X]$. We have $I^{-1} \neq R[X]$ by assumption. Moreover, the condition $(I : I) = R[X]$ implies that $c(I)_v = R$ by Proposition 1.6.

(5) $\Rightarrow$ (4): Trivial.

(3) $\Rightarrow$ (1): Proceeding contrapositively, suppose that R is not a UMV-domain, and let P be an upper to zero which is not maximal divisorial. Assume that $P^{-1} \neq R[X]$. Then $P \subsetneqq J$ for some divisorial ideal J of $R[X]$, and we must have $J \cap R \neq 0$. By Lemma 1.4, $c(J)_v \neq R$, whence $c(P)_v \neq R$. $\square$

We recall that a *v-domain* is a domain in which each finitely generated ideal is v-invertible. Examples include all completely integrally closed domains.

Theorem 2.3. *The following statements are equivalent for a domain R.*

(1) R is a v-domain

(2) R is an integrally closed UMV-domain.

(3) R is integrally closed, and every upper to zero in $R[X]$ is v-invertible.

(4) R is integrally closed and every upper to zero $P = fK[X] \cap R[X]$ with f a linear polynomial is v-invertible.

Proof. (2) $\Leftrightarrow$ (3): This follows from the fact that in the integrally closed case v-invertibility and maximal divisoriality are equivalent for uppers to zero (Proposition 1.8).

(1) $\Rightarrow$ (3) Suppose that R is a v-domain. It is well known that R is integrally closed. Let $P = fK[X] \cap R[X]$ be an upper to zero. Then $P = fc(f)^{-1}R[X]$ by [17, Lemme 1]. Since R is a v-domain, we have $(c(f)c(f)^{-1})_v = R$. Hence $(PP^{-1})_v = (c(f)^{-1}c(f)_v)_vR[X] = R[X]$, and P is v-invertible.

(2) $\Rightarrow$ (1) Assume that R is an integrally closed domain UMV-domain, and let A be a finitely generated ideal of R. Since a principal ideal is trivially v-invertible, we may as well assume that A is not principal. Then $A = c(f)$ for some nonconstant polynomial $f \in R[X]$. Set $I = fK[X] \cap R[X]$. Then $I = fc(f)^{-1}R[X]$ by [17, Lemme 1]. By Theorem 2.2, $(I : I) = R[X]$. Hence by Proposition 1.6, I is v-invertible in $R[X]$. It follows easily that $c(f) = A$ is v-invertible in R. Hence R is a v-domain.

(3) $\Rightarrow$ (4): Trivial.

(4) $\Rightarrow$ (1): Assume (4). To show that R is a v-domain, it suffices by [14, Lemma 3.6] to show that every two-generated ideal of R is v-invertible. Consider a two-generated ideal $A = (a, b)$. Let $f = aX + b$, and set $P = fK[X] \cap R[X]$. By hypothesis P is v-invertible. Again using [17, Lemme 1], we can write $P = fc(f)^{-1}R[X]$, and it is easy to see that $c(f) = A$ is v-invertible in R. □

We now compare our results on UMV-domains to those on UMT-domains. Recall that a domain R is a *UMT-domain* if each upper to zero in $R[X]$ is a maximal t-ideal, and that R is a *Prüfer v-multiplication domain* (PVMD) if each finitely generated ideal of R is t-invertible. According to [12, Proposition 3.2], a domain is a PVMD if and only if it is an integrally closed UMT-domain. Our Theorem 2.3 above provides an exact analogue to this PVMD characterization. On the other hand, our Theorem 2.2 is much less satisfying than the corresponding situation for UMT-domains. Recall from [12, Proposition 1.1] that if R is a domain and M is a maximal t-ideal of $R[X]$ with $M \cap R \neq 0$, then $M = (M \cap R)R[X]$. Hence, since in general a t-ideal of a domain is contained in a maximal t-ideal, we have that a domain R fails to be a UMT-domain if and only if there is an upper to zero P in $R[X]$ and a maximal t-ideal M of R with $P \subseteq MR[X]$. Unfortunately, however, maximal divisorial ideals need not exist. (For example, let R be a valuation domain whose maximal ideal is not principal.) The closest that we can come to an analogue of the UMT-result is the following reworking of the statement of Theorem 2.2 (1) $\Leftrightarrow$ (3).

Proposition 2.4. *Let R be a domain. Then R fails to be a UMV-domain if and only if there is an upper to zero P in $R[X]$ for which either $P^{-1} = R[X]$ or $P \subseteq IR[X]$ for some divisorial ideal I of R.*

Proof. Just take $I = c(P)_v$ is the statement of Theorem 2.2 (3). □

In [5, Example 3.1], Gabelli and Roitman give an example of a completely integrally closed (and therefore a v-) domain R admitting an upper to zero which is a maximal divisorial ideal but not a maximal t-ideal. In view of Theorem 2.3 (and the characterization of PVMDs as integrally closed UMT-domains), any v-domain which is not a PVMD must admit such an upper to zero. The first non-PVMD v-domain was produced by Dieudonné in [2]; other examples were given in [15, 16], [9], and [8]. Now there is a host of such examples. For example, if R is such an example, then so is $R[X]$ (as is mentioned in the next section below, if R is a v-domain, then $R[X]$ is also a v-domain). Examples can also be produced using the $D + XD_S[X]$-construction; see [18]. Thus, from our point of view, [5, Example 3.1] is another, welcome, addition to the list of v-domains which are not PVMDs.

By the characterizations mentioned above, a v-domain which is not a PVMD is a UMV-domain which is not a UMT-domain. It is of interest to have non-integrally closed examples of this phenomenon. We now give such an example.

Example 2.5. An example of a non-integrally closed UMV-domain which is not a UMT-domain. Begin with any domain D such that D is a v-domain but is not a PVMD, let k denote the quotient field, let $T = k[[t^2, t^3]]$ (t an indeterminate), and let R be defined by the following pullback diagram.

$$\begin{tikzcd} R \arrow[r] \arrow[d] & D \arrow[d] \\ T = \arrow[r, "\phi"] & k = T/M. \end{tikzcd}$$

Note that t is integral over R, so R is not integrally closed. Also, since D is not a UMT-domain, neither is R [4, Theorem 3.7]. It remains to show that R is a UMV-domain. Accordingly, let $P = fK[X] \cap R[X]$ be an upper to zero in $R[X]$, and let $Q = fK[X] \cap T[X]$ (K is the common quotient field of R and T). Since T is Noetherian, Q is almost principal, and we have $uQ \subseteq fT[X]$ for some nonzero $u \in T$. Hence $t^2uP \subseteq fR[X]$. Thus P is almost principal and therefore divisorial. In particular, $P^{-1} \neq R[X]$. We next want to show that $c(P)_v = R$. Again, since T is Noetherian (and one dimensional), we have $Q \nsubseteq M[X]$. Therefore, since $R_M = T$, we have $P \nsubseteq M[X]$. Now suppose $c(P)_v \neq R$, and pick $y \in c(P)^{-1} \setminus R$. Then $P \subseteq (R :_R y)R[X]$, and since the ideals of R are comparable to M, we must have $M \subsetneqq (R :_R y)$. It is then easy to show that $\phi(P)$ is an upper to zero in $D[X]$ with $\phi(P) \subseteq (D :_D \phi(y))D[X]$. However, since D is a UMV-domain, this contradicts Theorem 2.2. Hence we have $c(P)_v = R$, as desired. Another application of Theorem 2.2 completes the proof.

According to [4, Theorem 1.5], a domain R is a UMT-domain if and only if R_Q has Prüfer integral closure for each prime t-ideal Q of R. We have the following partial analogue.

Theorem 2.6. *If R is a UMV-domain, then R_Q has Prüfer integral closure for each divisorial prime Q of R.*

Proof. Let R be a UMV-domain, and let Q be a divisorial prime of R. Since each upper to zero in $R[X]$ is maximal divisorial, we have $P \nsubseteq Q$ for each upper to zero P. Hence in $R_Q[X]$ the prime ideal $QR_Q[X]$ contains no uppers to zero. It follows that if T is the integral closure of R_Q and N is a prime of T lying over QR_Q, then $NT[X]$ contains no uppers to zero. We then have that T is a Prüfer domain by [6, Theorem 19.15]. $\square$

Corollary 2.7. *If R is a v-domain, then R_M is a valuation domain for each divisorial prime M of R.*

3 Questions

We close with several questions and comments related to the results of the first two sections.

Question 3.1. If R_Q has Prüfer integral closure for each divisorial prime Q of R, is R necessarily a UMV-domain? In view of Theorem 2.6, a positive answer to this question would yield a characterization of UMV-domains comparable to the characterization of UMT-domains mentioned just before the statement of Theorem 2.6.

It is known that if R is a UMT-domain (respectively, a v-domain), then so is $R[X]$ [4, Theorem 2.4] (respectively, [1, Corollary 1.6]). This motivates our next question.

Question 3.2. If R is a UMV-domain, is $R[X]$ also a UMV-domain? Here is a related question: If R is a UMV-domain, does it follow that if P is a prime ideal of $R[X, Y]$ such that P has height one and satisfies $P \cap R = 0$, then P is a maximal divisorial ideal of $R[X, Y]$?

If Question 3.1 has a negative answer, then the following question becomes interesting.

Question 3.3. If R has the property that R_Q has Prüfer integral closure for each divisorial prime Q of R, does $R[X]$ have this same property?

Recall that for uppers to zero t-maximality and t-invertibility are equivalent [12, Theorem 1.4]. However, in Example 1.10, we produced a domain R admitting an upper to zero P with $P^{-1} = R[X]$ (so that P is trivially v-invertible but is not (maximal) divisorial).

Question 3.4. Is there an example of a non-UMV-domain R with the property that every upper to zero in $R[X]$ is v-invertible? We do not know whether Example 1.10 is such an example. At any rate no such example can be integrally closed, by Proposition 1.8. Assuming the existence of such examples, one could ask whether the property that all uppers to zero are v-invertible would extend to the polynomial ring.

Question 3.5. If R has the property that each upper to zero $P = fK[X] \cap R[X]$ with f linear is maximal divisorial, is R necessarily a UMV-domain? Note that this is proved in the integrally closed case in Theorem 2.3.

The corresponding question for the UMT-property has an affirmative answer. Since this has not (to our knowledge) appeared in the literature, we state it formally and sketch a proof.

Proposition 3.6. *Suppose that R has the property that every upper to zero $P = fK[X] \cap R[X]$ with f linear is a maximal t-ideal. Then R is a UMT-domain.*

Proof. In a manner similar to that used in the proof of Theorem 2.6, one can show that for each maximal t-ideal M of R, we have that $NT[X]$ contains no linear uppers to zero, where T is the integral closure of R_M and N is a prime of T lying over MR_M. Hence for $u \in K$, we have that $P = (x-u)K[X] \cap T[X] \nsubseteq N[X]$ for each maximal ideal N of T. Thus u is the root of a polynomial not contained in $N[X]$. By the u, u^{-1}-lemma [6, Lemma 19.14], either u or u^{-1} lies in T_N. It follows that T is a Prüfer domain. Hence R is a UMT-domain by [4, Theorem 1.5]. □

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