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ON A PROPERTY OF PRE-SCHREIER DOMAINS

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Let D be a commutative integral domain with 1. An element $x \in D$ is called <u>primal</u> if $x|ab$ implies that $x = rs$ where $r|a$ and $s|b$. An integrally closed integral domain D is called a <u>Schreier</u> domain if every element of D is primal. We call D <u>pre-Schreier</u> if every element of D is primal. Schreier domains were introduced by Cohn [5] and what we call pre-Schreier domains, have featured in [5] and in several other papers (cf. e.g. [6] and [13]). Along with several other characterizations of pre-Schreier domains we show that D is a pre-Schreier domain if and only if for $a_1,a_2,b_1,b_2 \in D$, $a_ib_j|x$ implies that $x = rs$ where $a_i|r$ and $b_j|s$ for $i,j = 1,2$. (This gives us yet another characterization of UFD's as : D is a UFD if and only if (a) every element of D is a product of finitely many irreducible elements and (b) for a_1,a_2 ; $b_1,b_2 \in D$, $a_ib_j|x$ implies that $x = rs$ where $a_i|r$ and $b_j|s$). In the course of establishing this characterization we show that pre-Schreier domains have the following property.

(*). For $a_i, b_j \in D$ $i = 1,\ldots,m$ and $j = 1,\ldots,n$

$$(\bigcap_i (a_i))(\bigcap_i (b_j)) = \bigcap_{i,j} (a_i b_j).$$

If we call an integral domain satisfying (*) a *-domain then a pre-Schreier domain is a *-domain but not conversely. This property is interesting in that it is an extremely weak property shared by integral domains which do not share the pre-Schreier property. For example, a Prüfer domain is a *-domain and so is a Bezout domain whereas according to [5] (Th. 2.8) a Prüfer domain which is also pre-Scheier is a Bezout domain. Weak though this property is, its presence gives the ring a specific character. For example a Krull domain which is also a *-domain is locally factorial. In fact, as Anderson et al [2] show (Corollary 3.9), a still weaker property

(**) $a_1, a_2 ; b_1, b_2 \in D$ implies $((a_1) \cap (a_2))((b_1) \cap (b_2)) = \cap(a_i b_j)$

makes a noetherian domain into a locally factorial domain. This in fact means that in some situations this rather mild property is equivalent to the stronger properties : the pre-Schreier property and / or the GCD-property.

In the course of establishing the above, we also show that if D is pre-Schreier and if S is a multiplicative set in D then D_S is again pre-Schreier. We also give a simple example of a strictly pre-Schreier domain and illustrate using a simple computation, that if D is strictly pre-Schreier then D[X] is not pre-Schreier where X is an indeterminate over D. Finally the notation and terminology used in this article comes mainly from [5] and [10], and we use the letter D to denote a commutative integral domain with 1 and we use K for the quotient field of D.

Apart from the introduction, this article is split into four sections. In the first section we present several characterizations of pre-Schreier domains and show that a pre-Schreier domain has the *-property. In section 2 we characterize the *-domains and by linking the various characterizations of pre-Schreier domains with those of Riesz groups we establish that the *-property emerges from a discrepancy between the definition of product of complexes in groups and that of the product of ideals in rings. Also included in this section is a comparison of the *-domains with the condensed domains of [3]. In section 3, we establish a method of translating some results on Prüfer domains to results on some generalizations of Prüfer domains. We use this method to study the effects of the *-property on the integral domains known as Prüfer v-multiplication domains (PVMD's). We show that a PVMD has the *-property if and only if it is locally a GCD-domain. From this it follows that a quasi-local PVMD (Krull domain) is a GCD-domain(UFD) if and only if it is a *-domain. In section 4, we show that in a pseudo valuation domain the *-property is equivalent to the pre-Schreier property. We use the observations made in this section to construct a simple example of a pre-Schreier domain which is not integrally closed and give computed evidence of the fact that if D is strictly pre-Schreier then a ring of polynomials over D is not pre-Schreier.

Finally, this article is a revised version of an earlier unpublished note, "On pre-Schreier domains" (cf. [2] and [3]).

1. Pre-Schreier domains and the *-property.

THEOREM 1.1. In D the following are equivalent.

(1) D is a pre-Schreier domain.

(2) For all $a,b \in D^* = D - \{0\}$ and for all $x,y \in (a) \cap (b)$ there exists $d \in (a) \cap (b)$ such that $d|x$ and $d|y$.

(3) For all $a,b \in D^*$ and for all $x_1,\dots,x_n \in (a) \cap (b)$ there exists $d \in (a) \cap (b)$ such that $d|x_i$; $i = 1,\dots,n$.

(4) For all $a_1,\dots,a_n \in D^*$ and $b_1,\dots,b_m \in \cap (a_i)$ there exists $d \in \cap (a_i)$ such that $d|b_j$; $j = 1,\dots,m$.

PROOF. (1) ⇒ (2). Let D be pre-Schreier, let $a,b \in D^*$ and let $x,y \in (a) \cap (b)$. Then $x = x_1a = x_2b$ and

$$y = y_1a = y_2b .$$

Since $x_1a = x_2b$ we have $b|x_1a$ and by the pre-Schreier property $b = b_1b_2$ where $b_1|x_1$ and $b_2|a$. Let $x_1 = x_1'b_1$ and $a = a_1b_2$. Then $x_1a = x_2b$ can be written as $x_1'b_1a_1b_2 = x_2b$ which, on cancelling b from both sides gives $x_1'a_1 = x_2$. Since $a_1b_2 = a$ we have $a,b|a_1b$ and using the value of x_2 we have that $a_1b|x$. Now consider $y_1a = y_2b$. Using $a = a_1b_2$ and $b = b_1b_2$ we have $y_1a_1b_2 = y_2b_1b_2$. Cancelling b_2 from both sides we get $y_1a_1 = y_2b_1$. So that $b_1|y_1a_1$ and by the pre-Schreier property $b_1 = b_3b_4$ where $b_3|y_1$ and $b_4|a_1$. Writing $y_1 = y_1'b_3$ and $a_1 = a_1'b_4$ we can express $y_1a_1 = y_2b_1$ as $y_1'b_3a_1'b_4 = y_2b_1$. Cancelling $b_1 = b_3b_4$ from both sides we get $y_2 = y_1'a_1'$. This gives $y = y_2b = y_1'a_1'b = y_1a$. Now as $y_1'|y_1$ we get $a|a_1'b$. That is $a,b|a_1'b$ and $a_1'b|y$. But as $a_1'|a_1$ we have $a_1'b|x$ also. So we have $d = a_1'b \in (a) \cap (b)$ such that $d|x$ and $d|y$.

(2) ⇒ (1). Let $a|bc$. Then as $bc, ab \in (a) \cap (b)$ there exists

$x \in (a) \cap (b)$ such that $x|ab$ and $x|bc$ (i)

Now as $x \in (a)$ $x = x_1a$ .... (ii)

and as $x \in (b)$ $x = x_2b$ .... (iii).

Using (i) and (iii) $x_2|a$ and $x_2|c$.

Now as $x_2|a$ we have, from $x_1a = x_2b$, $b = x_1(a/x_2)$. So $a = x_2(a/x_2)$ where $x_2, a/x_2 \in D$, $a/x_2|b$ and $x_2|c$.

(2) ⇒ (3). Let $x_1,\dots,x_n \in (a) \cap (b)$. If $n = 2$ we are through. So suppose that $n > 2$ and suppose that for all $x_1,\dots,x_{n-1} \in (a) \cap (b)$ the statement is true. Then for $x_1,\dots,x_{n-1} \in (a) \cap (b)$ there exists $d_1 \in (a) \cap (b)$ such that $d_1|x_i$ $(i = 1,\dots,n-1)$. But then for $d_1, x_n \in (a) \cap (b)$ there exists $d|d_1,x_n$ and so $d|x_i$ $(i = 1,\dots,n)$.

(3) ⇒ (4). Let $b_1,\dots,b_m \in \cap(a_i)$. Then $b_j \in (a_1) \cap (a_2)$ and so there is $d_1 \in (a_1) \cap (a_2)$ such that $d_1|b_j$; $j = 1,\dots,m$. Now

$b_1,\ldots,b_m \in (d_1) \cap (a_3) \cap ,\ldots, \cap (a_n)$ and induction on n completes the job.

(4) $\Rightarrow$ (2). Obvious.

The proof of the equivalence (1) $\Longleftrightarrow$ (2) gives rise to another characterization of the pre-Scheier domains.

COROLLARY 1.2. An integral domain D is a pre-Schreier domain if and only if for all $x,y \in D^*$ and for all $h \in (x) \cap (y)$ there exists $k \in (x) \cap (y)$ such that $k|h,xy$.

A module M is said to be <u>locally cyclic</u> if every finitely generated submodule of M is contained in a cyclic submodule of M. Thus D is a pre-Schreier domain if and only if every finite intersection of principal ideals of D is locally cyclic. It is routine to check that if M is a locally cyclic D-module then for a multiplicative set S of D, M_S is a locally cyclic D_S-module.

COROLLARY 1.3. If D is a pre-Schreier domain and S is a multiplicative set in D then D_S is a pre-Schreier domain.

PROOF. Let $A = \cap d_i D_S$ where $d_i = a_i/s_i$, $a_i \in D^*$ and $s_i \in S$. Then $A = \cap a_i D_S = (\cap(a_i))D_S$ and $\cap (a_i)$ is locally cyclic.

COROLLARY 1.4. An integral domain D is a pre-Schreier domain if and only if every finite intersection of principal fractional ideals is locally cyclic.

PROOF. The if part is an obvious application of Theorem 1.1. For the only if part let D be pre-Schreier and let $M = \cap(x_i)$. Then if $x_i = a_i/b_i$ putting $b = \prod_{i=1}^{n} b_i$ and $B_i = b_1\ldots\hat{b}_i\ldots b_n$ where $\hat{b}_i = 1$ we get $M = (1/b)\left(\bigcap_{i=1}^{n} (a_iB_i)\right)$.

Now $\cap (a_iB_i)$ is locally cyclic and if $z_i \in M$ $(i = 1,\ldots,m)$. Then $z_i = y_i/b$ where $y_i \in \cap (a_iB_i)$. But then by Theorem 1.1, there exists $d_1 \in \cap (a_iB_i)$ such that $d_1|y_i$ and clearly

$(z_i) = (y_i/b) \subseteq (d_1/b)$. So we have $d = d_1/b \in M$ such that $(z_1,\ldots,z_m) \subseteq (d)$.

COROLLARY 1.5 (Cf. Lemma 1 of [13]). For D the following are equivalent.

(1) D is pre-Schreier.

(2) If $b \in K$ then $(1,b)^{-1}$ is locally cyclic.

(3) The inverse of every finitely generated fractional ideal is locally cyclic.

PROOF. The proof follows from the fact that the inverse of every finitely generated fractional ideal is an intersection of the inverses of the generators.

The above results give us the tool needed for the promised characterization of pre-Schreier domains.

THEOREM 1.6. An integral domain D is a pre-Schreier domain if and only if for all $a_1,\ldots,a_n$; $b_1,\ldots,b_m \in D^*$ and for all $x \in D$ with $a_ib_j | x$ (for all $i = 1,\ldots,n$; $j = 1,\ldots,m$) we have $x = rs$ where $a_i | r$ and $b_j | s$.

PROOF. Let $a,b \in D^*$ and let $c,d \in (a) \cap (b)$. Then since each of a,b divides each of c,d we have $ac,ad,bc,bd | cd$. Now by the hypothesis $cd = rs$ where $a,b | r$ and $c,d | s = cd/r$. Now $c | cd/r$ and so $rc | cd$ which gives $r | d$. Further $d | cd/r$ and so $rd | cd$ which gives $r | c$. Consequently we have $r \in (a) \cap (b)$ such that $r | c,d$. This by (2) of Theorem 1.1 proves that D is a pre-Schreier domain.

Conversely suppose that D is a pre-Schreier domain and let $a_i,b_j \in D^*$. If $a_ib_j | x$ (for all $i = 1,\ldots,n$, $j = 1,\ldots,m$) then $b_j | x/a_1,\ldots,b_j | x/a_n$ for all j and this gives $x/a_1,\ldots,x/a_n \in \cap (b_j)$. But then by (4) of Theorem 1.1 there exists $s \in \cap (b_j)$ such that $s | x/a_i$. Thus $x = (x/s)s$ where $a_i | x/s$ and $b_j | s$. Now putting $x/s = r$ we get the result.

COROLLARY 1.7. The following hold.

(1) A pre-Schreier domain is a *-domain.

(2) An integral domain D is a pre-Schreier domain if and only if for $a_1, a_2 ; b_1, b_2 \in D^*$, $a_i b_j | x$ implies $x = rs$ where $a_i | r$ and $b_j | s$.

PROOF.

(1). We note that for $a_i, b_j \in D$, $(\cap(a_i))(\cap(b_j)) \subseteq \cap (a_i b_i)$ $(i = 1,\ldots,n ; j = 1,\ldots,m)$. Now let $x \in \cap (a_i b_j)$ then by Theorem 1.6, $x = rs$ where $r \in \cap (a_i)$ and $s \in \cap (b_j)$ and so $\cap (a_i b_j) \subseteq (\cap(a_i))(\cap(b_j))$.

(2). The sufficiency is proved in Theorem 1.6 and the necessity is a special case of the necessity in Theorem 1.6.

Of interest may be the fact that Theorem 1.6, affords a new characterization of UFD's.

According to Cohn [5] (Th. 5.3) an atom (irreducible element) in a Schreier domain is a prime and the same conclusion obsviously holds in a pre-Schreier domain. So if we call an integral domain _atomic_ when its elements are expressible as products of finitely many atoms (each) we have the following corollary.

COROLLARY 1.8. An integral domain D is a UFD if and only if (a) D is atomic and (b) for all $a_1, a_2 ; b_1, b_2 \in D^*$ and for all x with $a_i b_j | x$ we have $x = rs$ where $a_i | r$ and $b_j | s$.

2. A characterization and the origins of the *-domains.

From Theorem 1.6 and Corollary 1.7 it follows that a pre-Schreier domain is a *-domain with the special property that if $x \in (\cap(a_i))(\cap(b_j))$ then $x = rs$ where $r \in \cap (a_i)$ and $s \in \cap (b_j)$. This special property indeed is the strength of the pre-Schreier property. We now proceed to show, indirectly, that a *-domain is not necessarily a pre-Schreier domain. For this purpose we show that the *-property is locally characterizable. As usual we call D locally X if for each maximal ideal M, D_M has property X.

THEOREM 2.1. An integral domain D is a *-domain if and only if it is locally a *-domain.

PROOF. The very definition of a *-domain indicates that if D is a *-domain and if S is a multiplicative set in D then D_S is a *-domain. So the condition is necessary.

Conversely let for each maximal ideal M, D_M be a *-domain and let $a_i, b_j \in D$. Then $(\cap(a_i))(\cap(b_j)) = \cap(\cap(a_i))(\cap(b_j))D_M$ where M ranges over all the maximal ideals of D. But since each of D_M is a *-domain $(\cap(a_i))(\cap(b_j))D_M = (\cap(a_i))D_M(\cap(b_j))D_M = (\cap a_iD_M)(\cap b_jD_M)$

$$= \cap a_ib_jD_M = (\cap(a_ib_j))D_M$$

and hence $(\cap(a_i))(\cap(b_j)) = \cap(\cap(a_ib_j))D_M = \cap(a_ib_j)$.

Now we have established that a pre-Schreier domain is a *-domain and we know that a GCD-domain is pre-Schreier (cf. [5]). So by Theorem 2.1 if D is locally pre-schreier, GCD or Bezout then it is a *-domain. Because a Prüfer domain D has the property that for each maximal ideal M, D_M is a valuation domain we have the following corollary.

COROLLARY 2.2. A Prüfer domain is a *-domain.

That a *-domain is not necessarily a pre-Schreier domain follows from the fact that a Prüfer domain which is also a Schreier domain is Bezout ([5] (Th. 2.8)) and obviously not every Prüfer domain is Bezout.

NOTE 2.3. It is easy to see that for D the following are equivalent.

(1) D has the *-property.

(2) For all $x_i, y_j \in K - \{0\}$ where $i = 1,\ldots,n$ and $j = 1,\ldots,m$ we have $(\cap(x_i))(\cap(y_j)) = \cap(x_iy_j)$.

Combining this note with the rather direct characterization of *-domains we get the following theorem.

THEOREM 2.4. In an integral domain D the following are equivalent.

(1) D is a *-domain.

(2) For all invertible ideals $A_1,\dots,A_n$; $B_1,\dots,B_m$ of D $(\cap A_i)(\cap B_j) = \cap A_iB_j$.

PROOF. (2) ⇒ (1) is obvious because every non-zero principal fractional ideal is invertible. So we prove (1) ⇒ (2).

Let D be a *-domain. Then we note that :

(i) D_M is a *-domain for every maximal ideal M,

(ii) in a quasi-local domain every invertible ideal is principal,

and

(iii) if A is invertible then AD_S is invertible for any multiplicative set S in D.

Now let $A_1,\dots,A_n$; $B_1,\dots,B_m$ be invertible ideals of D. Then

$$(\cap A_i)(\cap B_j) = \cap\,((\cap A_i)(\cap B_j))D_M \quad \text{(where M ranges over maximal ideals)}$$
$$= \cap\,(\cap A_iD_M)(\cap B_jD_M)$$
$$= \cap\,(\cap A_iD_MB_jD_M)$$

(since A_iD_M, B_jD_M are principal and D_M are *-domains)

$$= \cap\,(\cap A_iB_j)D_M = \cap A_iB_j\,.$$

Now as a Prüfer domain is a *-domain it satisfies (2) of Theorem 2.4. Further, because in a Prüfer domain every finitely generated ideal is invertible we can say that in a Prüfer domain for finitely generated fractional ideals $A_1,\dots,A_n$; $B_1,\dots,B_m$ $(\cap A_i)(\cap B_j) = \cap A_iB_j$.

Conversely if for all finitely generated fractional ideals A_i, B_j of D, $(\cap A_i)(\cap B_j) = \cap A_iB_j$ $(i \leq n,\ j \leq m)$ then by (c_1) of

Theorem 25.2 of [10], D is a Prüfer domain. This gives us the following corollary.

COROLLARY 2.5. An integral domain D is a Prüfer domain if and only if for all finitely generated fractional ideals A_i, B_j $(1 \le i \le n,\ 1 \le j \le m)$, $(\cap A_i)(\cap B_j) = \cap A_iB_j$.

This corollary gives us a rather interesting result.

COROLLARY 2.6. An integral domain D is a Bezout domain if and only if for all finitely generated integral ideals A_i, B_j $(1 \le i \le 2;\ 1 \le j \le 2)$ $r \in \cap A_iB_j$ implies that $x = rs$ where $r \in \cap A_i$ and $s \in \cap B_j$.

PROOF. Suppose that D is Bezout. Then each of A_i, B_j is principal and as a Bezout domain is also pre-Schreier the result follows from Theorem 1.6. Conversely suppose that the condition holds. As it also holds for principal ideals we have from (2) of Corollary 1.7, that D is a pre-Schreier domain. Now we show that in general the condition implies that $(\cap A_i)(\cap B_j) = \cap A_iB_j$. By (c_1) of 25.2 of [10] this is sufficient for D to be a Prüfer domain, and this will complete the proof.

Let $x \in (\cap A_i)(\cap B_j)$. Then $x = \sum_{k=1}^{r} u_kv_k$ where $u_k \in \cap A_i$ and $v_k \in \cap B_j$. Consequently $u_k \in A_i$ for all i and $v_k \in B_j$ for all j. Thus $u_kv_k \in A_iB_j$ for all i,j and from this we conclude that $x = \sum u_kv_k \in \cap A_iB_j$ and that $(\cap A_i)(\cap B_j) \subseteq \cap A_iB_j$. As the reverse inclusion is given by the condition, we have the equality.

REMARK 2.7. (The origin of the *-domains).

A primal element is said to be <u>completely primal</u> if each of its factors is primal. Cohn [5] (Lemma 2.5) shows that the product of two completely primal elements is completely primal. From his work (loc cit) one can easily derive the result that a multiplicative set generated by completely primal elements has the Riesz refinement property, that is if $x = a_1 \ldots a_n = b_1 \ldots b_m$ where each of a_i, b_j is completely primal then the two factorizations of x have a common refinement i.e. there are elements c_{ij} such that $a_i = \prod_{j=1}^{m} c_{ij}$, $b_j = \prod_{i=1}^{n} c_{ij}$.

From the above observations it follows that D is a pre-Schreier domain if and only if its group of divisibility G(D) is an abelian Riesz group (an abelian directed P.O. group whose elements satisfy the above mentioned refinement property). The reader may consult Fuchs [9] for the general definition and various characterizations of a Riesz group. We note that once the connection of pre-Schreier domains and Riesz groups is clear, one can easily translate various characterizations of Riesz groups to those of pre-Schreier domains. We further note that (2) and (4) of Theorem 1.1 are what Fuchs (loc cit) calls the (2,2) and the (m,n) interpolation properties respectively. In the p.o. group terminology our statements are made for positive elements only and hence are slightly stronger (Corollary 1.4 incidentally is a true restatement of the (m,n) interpolation property mentioned by Fuchs (loc cit). From Corollary 1.7 it follows that D is a pre-Schreier domain if and only if it is a *-domain with the property that for $x \in (\cap(a_i))(\cap b_j))$, $x = rs$ where $r \in \cap(a_i)$ and $s \in \cap(b_j)$. This too is a translation of a known characterization of Riesz groups. Yet we arrive at this translation by bridging a gap. To see this gap we note that if G(D) is the group of divisibility of D and if $x_1, \ldots, x_n \in G(D)$ then the set of all upper bounds of $\{x_1, \ldots, x_n\}$ which is $U(x_1, \ldots, X_n) = \{t \in G(D) \mid t \geq x_i\}$ is represented in ring-theoretic terms by $\cap x_i D$. By Theorem 2.2 (part (3)) of [9], G(D) is a Riesz group if and only if for all $x_1, \ldots, x_n ; y_1, \ldots, y_m \in G(D)$ $U(x_1, \ldots, x_n)U(y_1, \ldots, y_m) = U(x_1 y_1, \ldots, x_i y_j, \ldots, x_n y_m)$. We note that the product here means $AB = \{ab \mid a \in A \text{ and } b \in B\}$. On the other hand the product of two ideals HK is given by

$HK = \{ \Sigma h_i k_i \mid h_i \in H \text{ and } k_i \in K\}$. That is if H and K are ideals of D and if $x \in HK$ it is not necessary that $x = rs$ where $r \in H$ and $s \in K$. So $(\cap x_i D)(\cap y_j D) = \cap x_i y_j D$ is not sufficient, ring theoretically, to imply that D is a pre-Schreier domain. Thus the difference between the definitions of products of subsystems is solely responsible for the notion of *-domains.

It would be unfair not to mention at this point the beautiful paper [3] which discusses integral domains D for any two ideals A,B of which $x \in AB$ implies that $x = ab$ where $a \in A$ and $b \in B$. In [3] these integral domains are called condensed. One may suspect that a condensed domain is a special case of a pre-Schreier domain, but this is not so. The reason is that being a pre-Schreier domain involves two conditions whereas being a condensed domain involves only one. To elaborate on this point we take up an example of a condensed domain and show that it does not have the *-property.

EXAMPLE 2.8 (Cf. [3] Example 2.3). Let F be a field, let R be the ring of those power series (over F) whose coefficient of X is zero. Then according to [3] R ($= F[[X^2,X^3]]$) is a condensed domain. We show that R is not a *-domain and hence is not a pre-Schreier domain.

Let $A = (X^2) \cap (X^3)$ and $B = (X^3) \cap (X^4)$. Then $A = (X^5, X^6)$ and $B = (X^6, X^7)$. This follows from the fact that every non-principal ideal of R is of the type (X^n, X^{n+1}) $n \geq 2$ (Cf. [4] Example 1(a) p. 545).

$$\text{So } AB = ((X^2) \cap (X^3))((X^3) \cap (X^4)) = (X^5, X^6)(X^6, X^7)$$
$$= (X^{11}, X^{12}, X^{13}) = (X^{11}, X^{12}).$$

Now if we put $X^2 = a$, $X^3 = b$, $X^3 = c$ and $X^4 = d$ we have

$$AB = ((a) \cap (b))((c) \cap (d)) \quad \text{whereas}$$
$$C = (ac) \cap (ad) \cap (bc) \cap (bd) = (X^5) \cap (X^6) \cap (X^6) \cap (X^7)$$
$$= (X^6) \cap (X^7) = (X^9, X^{10}).$$

Thus $AB \subset C$ and so R is not a *-domain.

The readers who are not interested in number crunching of the sort done above may take the following alternative route. Obviously $R = F[[X^2, X^3]]$ is noetherian and according to [2] (Cor. 3.9) a noetherian domain which satisfies * is locally factorial. The result then follows from the fact that R is not factorial.

The above example, as the referee has pointed out, can be used to draw an interesting conclusion. We first note, by way of preparation, that all the non-zero ideals of $F[[X^2, X^3]]$ are divisorial (Cf. [10] Example 11, p. 431). So, according to Example 2.8, there may exist in some integral domain D elements a_i, b_j such that $(\bigcap_i (a_i))(\bigcap_j (b_j))$ is divisorial without being equal to $\bigcap_{i,j} (a_i b_j)$. In other words, for D to a *-domain it is necessary that products of the type $(\bigcap_i (a_i))(\bigcap_j (b_j))$ be divisorial, but it is not sufficient. In the next section (Corollary 3.3) we shall indicate the situation in which this condition is also sufficient to make D a *-domain.

3. The *-property and some generalizations of Prüfer domains.

An integral domain D is said to be <u>essential</u> if it has a family $\{P_i\}_{i \in I}$ of prime ideals such that D_{P_i} is a valuation domain for each i and $D = \bigcap D_{P_i}$. So an essential domain is a generalization of a Prüfer domain in that, like a Prüfer domain it is an intersection of valuation <u>overrings</u> (rings between D and K) which are localizations of it. PVMD's (introduced below) are a special case of essential domains and hence are in the class of generalizations of Prüfer domains.

The aim of this section is to study how near to being *-domains are the essential domains and the PVMD's. We also study the conditions under which these integral domains are *-domains.

To introduce PVMD's and to introduce some of the notions we shall use in this connection we recall some of the related definitions and results from [10] (section 32 and 34 mainly). Let $F(D)$ be the set of (non-zero) fractional ideals of D. A mapping $A \to A^*$ of $F(D)$ into itself is called a *-operation on D if for all $a \in K - \{0\}$ and for all $A,B \in F(D)$

(i) $(a)^* = (a)$.

(ii) $A \subseteq A^*$ and $A \subseteq B$ implies $A^* \subseteq B^*$.

(iii) $(A^*)^* = A^*$.

From this definition it follows that for $A_i \in F(D)$ if $\bigcap_i A_i \in F(D)$ then $(\cap A_i^*)^* = \cap A_i^*$ and if $\sum_i A_i \in F(D)$ then $(\sum_i A_i)^* = (\sum_i A_i^*)^*$

A fractional ideal is called a *-ideal if $A^* = A$ and a *-ideal of finite type if $A = B^*$ for B finitely generated. Further, if $\{D_i\}_{i \in I}$ is a family of overrings of D such that $D = \cap D_i$ then the map $A \to \cap AD_i$ is a *-operation which is said to be induced by $\{D_i\}$. The operation $A \to (A^{-1})^{-1} = A_v$ is another *-operation called the v-operation. It has the special property that for any *-operation', $A' \subseteq A_v$ and $A' \subseteq B'$ implies that $A_v \subseteq B_v$ and hence $A' = B'$ implies $A_v = B_v$. An integral domain D is called a Prüfer v-multiplication domain if the set $H(D)$ of v-ideals of finite type forms a group under the v-multiplication $(AB)_v = (A_vB)_v = (A_vB_v)_v$. According to Griffin [11], a PVMD is essential. Finally if every v-ideal of finite type of D is invertible then D is called a generalized GCD-domain (Cf. [1]).

The result that we need to prove is that if D is essential domain $((\cap(a_i))(\cap(b_j)))_v = \cap(a_ib_j)$ and we shall prove it via a more general theorem which gives a method of translating results known for Prüfer domains to results for essential domains. For this result we prepare as follows. If the fractional ideals $A_1,\ldots,A_n$ are combined by usual operations on fractional ideals we call a particular such combination an expression on the ideals and denote it by $F(A_1,\ldots,A_n)$. We call an expression $F(A_1,\ldots,A_n)$ extendable if for each multiplicative set S of D,

$F(A_1,\ldots,A_n)D_S = F(A_1D_S,\ldots,A_nD_S)$. Finally we say that $F(A_1,\ldots,A_n) = G(A_1,\ldots,A_n)$ is an <u>identity</u> on D if for all $A_1,\ldots,A_n \in F(D)$ the given equation holds.

THEOREM 3.1. Let $F(A_1,\ldots,A_n)$ and $G(A_1,\ldots,A_n)$ be extendable expressions on any essential domain D and let $F(A_1,\ldots,A_n) = G(A_1,\ldots,A_n)$ be an identity on any Prüfer domain D'. Then $(F(A_1,\ldots,A_n))_v = (G(A_1,\ldots,A_n))_v$ is an identity on any essential domain D.

PROOF. Let D be essential and let $\{P_i\}_{i\in I}$ be a family of prime ideals of D such that D_{P_i} are all valuation domains and $D = \cap D_{P_i}$. Let w be the *-operation induced by $\{D_{P_i}\}$. Then

$$\begin{aligned}(F(A_1,\ldots,A_n))_w &= \cap F(A_1,\ldots,A_n)D_{P_i} \\ &= \cap F(A_1D_{P_i},\ldots,A_nD_{P_i}) && \text{(since F is extendable)} \\ &= \cap G(A_1D_{P_i},\ldots,A_nD_{P_i}) && \text{(since } D_{P_i} \text{ are all Prüfer)} \\ &= \cap G(A_1,\ldots,A_n)D_{P_i} && \text{(since G is extendable)} \\ &= (G(A_1,\ldots,A_n))_w .\end{aligned}$$

But this means that $(F(A_1,\ldots,A_n))_v = (G(A_1,\ldots,A_n))_v$. Since F = G is an identity over Prüfer domains we have the result.

The following corollary indicates some applications of this theorem.

COROLLARY 3.2. In an essential domain the following hold.

(i) For all finitely generated fractional ideals A_i, B_j $(1 \le i \le n$ and $1 \le j \le m)$ $((\cap A_i)(\cap B_j))_v = (\cap A_iB_j)_v$.

(ii) For all principal fractional ideals $(a_1),\ldots,(a_n)$, $(b_1),\ldots,(b_n)$; $((\cap (a_i))(\cap (b_j)))_v = \cap (a_ib_j)$.

(iii) For all finitely generated fractional ideals A_i, $1 \le i \le r$ $((\cap A_i)^n)_v = (\cap A_i^n)_v$.

(iv) For all integral ideals A, B, C

$$(A \cap (B + C))_v = (A \cap B + A \cap C)_v.$$

(v) For all integral ideals A, B

$$((A + B)(A \cap B))_v = (AB)_v.$$

(vi) For all integral ideals A, B, C

$$(A(B \cap C))_v = (AB \cap AC)_v.$$

(vii) For all integral ideals A, B, C and for C finitely generated

$$((A + B):C)_v = (A : C + B : C)_v.$$

PROOF. The proofs of (i) and (ii) follow from Theorem 3.1, via Corollary 2.5. The proof of (iii) can be constructed using 15.5 part (a) of [10] and the above theorem, while for (iv) - (vii) 25.2 of [10] can be used in addition to the above theorem.

It would be interesting to see if any of the expressions in Corollary 3.2 actually characterize essential domains.

In addition to the connection Corollary 3.2 establishes between Prüfer and essential domains it leads to the precise conditions under which an essential domain is a *-domain.

COROLLARY 3.3. An essential domain is a *-domain if and only if the product of each pair $\cap (a_i)$, $\cap (b_j)$, of finite inter-sections of principal fractional ideals, is a v-ideal.

Now as the PVMD's are essential, the results proved for essential domains also hold for PVMD's. Yet in case of PVMD's we can go still deeper.

COROLLARY 3.4. Let D be a PVMD. Then the following are equivalent.

(1) D is locally a GCD-domain.

(2) D is a *-domain.

(3) In D the product of every two v-ideals of finite type is again a v-ideal of finite type.

(4) For every two elements $a,b \in D^*$, $(a) \cap (b)$ is invertible.

(5) For every pair of elements $f,g \in K[X] - \{0\}$ $(A_{fg})^{-1} = (A_fA_g)^{-1} = A_f^{-1}\ A_g^{-1}$ where A_f denotes the ideal generated by the coefficients of f.

(6) D is a generalized GCD-domains.

PROOF.

(1) ⇒ (2) is obvious and (2) ⇒ (3) follows from Corollary 3.3, and the fact that D is a PVMD.

(3) ⇒ (4). Let a,b be any two elements of D. By (3).

$$(a,b)_v((a) \cap (b)) \quad \text{is a v-ideal.}$$

That is

$$(a,b)_v((a) \cap (b)) = ((a,b)_v((a) \cap (b)))_v .$$

But as D is a PVMD$((a,b)_v((a) \cap (b)))_v = (ab)$ which gives $(a,b)_v((a) \cap (b)) = (ab)$ which means that $(a) \cap (b)$ is invertible.

(4) ⇒ (1). Obvious.

(2) ⇒ (5). We recall that the *-property can be stated for principal fractional ideals. Now

$$(A_fA_g)^{-1} = ((a_1,\dots,a_n)(b_1,\dots,b_m))^{-1}$$

where a_i , b_j are coefficients of f and g respectively. Thus

$$(A_fA_g)^{-1} = (\dots,a_ib_j,\dots)^{-1} = \cap\,(a_ib_j)^{-1} = \cap\,(1/a_i\,.\,1/b_j)$$
$$= (\cap\,(1/a_i))(\cap\,(1/b_j)) = A_f^{-1}A_g^{-1} \quad \text{(by (2))}.$$

Moreover, as a PVMD is integrally closed $(A_{fg})_v = (A_fA_g)_v$ (Cf.[10], Prop. 34.8) and so $(A_{fg})^{-1} = (A_fA_g)^{-1} = (A_f)^{-1}(A_g)^{-1}$.

(5) ⇒ (2). As we can select f and g such that $A_f = (1/a_1,\dots,1/a_n)$ and $A_g = (1/b_1,\dots,1/b_m)$ we can use (5)

to derive (2). Finally (6) ⇒ (4) is a result of [1] and this completes the proof of the corollary.

REMARK 3.5. Corollary 3.4 can be used to prove that a semi-quasi-local PVMD with the *-property is a GCD-domain. On the more practical side a semi-quasi-local Krull domain is a UFD if and only if it is a *-domain.

It is of course well known that a PVMD is a GCD-domain if and only if it is a Schreier domain. Yet most proofs of this statement are indirect or involved. Theorem 1.1, affords a direct proof and in the following we indicate the procedure.

THEOREM 3.6. Let D be a pre-Schreier domain and let $a_1,\dots,a_n \in D^*$. Then the following are equivalent.

(1) $\cap (a_i)$ is principal.
(2) $\cap (a_i)$ is invertible.
(3) $\cap (a_i)$ is finitely generated.
(4) $\cap (a_i)$ is a v-ideal of finite type.

PROOF. The implications (1) ⇒ (2) ⇒ (3) ⇒ (4) are obvious. So we prove (4) ⇒ (1). For this we note that $\cap (a_i)$ is a v-ideal for any D. If $\cap (a_i)$ is of finite type then there exist $x_1,\dots,x_r$ such that $(x_1,\dots,x_r)_v = \cap (a_i)$. But by (4) of Theorem 1.1 there exists $d \in \cap(a_i)$ such that $(x_1,\dots,x_r) \subseteq (d) \subseteq \cap(a_i)$. But then $\cap (a_i) = (x_1,\dots,x_r)_v \subseteq (d) \subseteq \cap(a_i)$.

Now because in a PVMD $\cap (a_i)$ is always of finite type we conclude that a PVMD is a GCD-domain if and only if it is Schreier. In fact a more general statement can be made and I am thankful to the referee for it.

COROLLARY 3.7. Let D be a pre-Schreier domain. Let A be a v-ideal of finite type with A^{-1} also of finite type. Then A is principal. Hence Pic(D) = 0.

PROOF. Let $A^{-1} = (b_1,\ldots,b_n)_v$. Then $A = A_v = (b_1^{-1}) \cap \ldots \cap (b_n^{-1})$. By Theorem 3.6, A is principal. The second statement is obvious once we note that Pic(D) = Inv(D)/P(D) where Inv(D) denotes the group of invertible ideals and P(D) the group of non-zero principal fractional ideals.

Anderson et al [2] have introduced an interesting variation of the *-property. They show that a noetherian integral domain D is locally a UFD if and only if it satisfies : (**) for all $a,b,c,d \in D^*$, $((a) \cap (b))((c) \cap (d)) = (ac) \cap (ad) \cap (bc) \cap (bd)$. This gives rise to the question , "Does (**) imply (*) ?" This question looks plausible in that using part (2) of Corollary 1.7 we can show that D is a pre-Schreier domain if and only if it satisfies (**) and $x \in ((a) \cap (b))((c) \cap (d))$ implies that $x = rs$ where $r \in (a) \cap (b)$ and $s \in (c) \cap (d)$. So for a pre-Schreier domain (**) is equivalent to (*). There is another area where the equivalence can be easily seen. We note that according to Griffin [12] (Prop. 17), every v-ideal of finite type is a v-ideal of type 2 in a ring of Krull type. Here a ring of Krull type D is a PVMD with a family of prime ideals $\{P_i\}_{i \in I}$ such that D_{P_i} are all valuation domains and each non-zero non-unit of D is in at most a finite number of P_i. So in a ring of Krull type $\cap (a_i) = (1/a_1,\ldots,1/a_n)^{-1} = ((x,y)_v)^{-1} = (1/x) \cap (1/y)$. These observations give rise to the following corollary.

COROLLARY 3.8. Let D be a PVMD in which every v-ideal of finite type is a v-ideal of type two. Then the following are equivalent for D.

(1) D is locally a GCD-domain.

(2) D is a *-domain.

(3) In D the product of every two v-ideals of finite type is again a v-ideal.

(4) For every two elements $a,b \in D^*$, $(a) \cap (b)$ is invertible.

(5) For every pair of elements $f,g \in K[X] - \{0\}$

$$(A_{fg})^{-1} = (A_f A_g)^{-1} = A_f^{-1} A_g^{-1}.$$

(6) D is a (**)-domain.

(7) For every pair of linear polynomials $f,g \in K[X] - \{0\}$

$$(A_{fg})^{-1} = (A_f A_g)^{-1} = A_f^{-1} A_g^{-1}.$$

Indeed there are some doubts too. In the above mentioned situations where (**) is equivalent to (*) there are some stringent accompanying conditions as well. For example there is noetherian condition under which (**) forces the noetherian domain to be a Krull domain, as shown in [2], for which (**) is equivalent to (*) by Corollary 3.8. For pre-Schreier domains (**) is coupled with something that actually forces the pre-Schreier property into being. Finally, being a ring of Krull type or satisfying the condition of Corollary 3.8 is in itself a stringent condition ; as it is not necessary that in a PVMD every v-ideal of finite type should be of type two ... not even in a Prüfer domain (Cf. [14] and references there).

We close this section with another observation due to the referee.

PROPOSITION 3.9. Let D be a *-domain. Let A be a v-ideal of finite type such that A^{-1} is also of finite type. Then A is invertible.

PROOF. Let $A = (c_1,\ldots,c_m)_v$ so that $A^{-1} = (c_1^{-1}) \cap \ldots \cap (c_m^{-1})$. Let $A^{-1} = (b_1,\ldots,b_n)_v$ so that $A = A_v = (b_1^{-1}) \cap \ldots \cap (b_n^{-1})$. Then $AA^{-1} = ((b_1^{-1}) \cap \ldots \cap (b_n^{-1}))((c_1^{-1}) \cap \ldots \cap (c_m^{-1})) = \cap (b_i^{-1} c_j^{-1})$. But each $c_j \in (b_i^{-1})$. That is $R \subseteq \cap (b_i^{-1} c_j^{-1})$. Hence $AA^{-1} \supseteq R$ and and this gives $AA^{-1} = R$.

COROLLARY 3.10. A *-domain D is a generalized GCD-domain if and only if for every finitely generated non-zero ideal A of D, A^{-1} is of finite type.

PROOF. If D is a generalized GCD-domain then it is a *-domain and a PVMD. So, for every finitely generated non-zero ideal A, A^{-1} is of finite type.

Conversely, using the fact that $A^{-1} = A_v^{-1}$ we conclude that every finite type v-ideal has the property that A^{-1} is of finite type. But then by Proposition 3.9, every v-ideal of finite type is invertible. Thus D is a PVMD and Corollary 3.4 applies.

4. The *-property and the pseudo valuation domains.

Recall that a quasi-local integral domain (D,M), with the maximal ideal M, is a pseudo valuation domain (PVD) if for all ideals A, B of D we have $A \subseteq B$ or $BM \subseteq AM$. Clearly if (D,M) is a PVD then $\{mM \mid m \in M\}$ is a chain under inclusion. Clearly if D is a PVD which is not a valuation domain then there exists at least one pair of elements $a,b \in D$ with aD and bD incomparable. But then $aM \subseteq bM$ or $bM \subseteq aM$. We assume that $aM \subseteq bM$. Then $aD \supseteq aD \cap bD \supseteq aM$. Since aM is a maximal submodule we have $aD \cap bD = aM$. Thus we have the following result.

PROPOSITION 4.1. Let (D,M) be a PVD. Then for any two incomparable principal ideals aD,bD of D, $M = aD : bD$ or $M = bD : aD$.

The following result follows immediately from Proposition 4.1.

COROLLARY 4.2. If (D,M) is a PVD which is not a valuation domain then :

(i) M is a v-ideal.

(ii) M is finitely generated if and only if D is a finite conductor domain (i.e. the intersection of every two principal ideals is finitely generated).

(iii) For all $a,b \in D$ one of the following holds : $a \mid b$, $b \mid a$, $aD \cap bD = aM$ or $aD \cap bD = bM$.

The proofs are obvious.

COROLLARY 4.3. Let (D,M) be a PVD then the following are equivalent.

(i) M is a locally cyclic D-module.

(ii) For all $a,b \in D$; $aD \cap bD$ is locally cyclic.

(iii) D is a pre-Schreier domain.

PROOF.

(i) ⇒ (ii). According to (iii) of Corollary 4.2 either $aD \cap bD = xD$ or $aD \cap bD = xM$ where $x = a$ or $x = b$. If $aD \cap bD = xD$ then $aD \cap bD$ is cyclic and hence locally cyclic. Further, for $x \in D$ xM is locally cyclic if and only if M is. So if M is locally cyclic then so is $aD \cap bD = xM$.

(ii) ⇒ (iii). This follows from (3) of Theorem 1.1

(iii) ⇒ (i). If D is a valuation domain then clearly M is locally cyclic ; as every finitely generated ideal is principal in a valuation domain. If D is not a valuation domain then by Proposition 4.1, $M = (a) : (b)$ for some $a,b \in D$. But $(a) : (b)$ is locally cyclic because of the pre-Schreier property.

We now use these results to show that in a PVD which is not a valuation domain the *-property is equivalent to the pre-Schreier property and to many other stronger properties.

THEOREM 4.4. Let (D,M) be a PVD which is not a valuation domain. Then the following are equivalent :

(i) D is a pre-Schreier domain.

(ii) D is a *-domain.

(iii) $M^2 = M$.

(iv) M is a flat D-module.

(v) M is a locally cyclic D-module.

(vi) D is a (**)-domain.

PROOF.

(i) ⇒ (ii). This is (1) of Corollary 1.7

(ii) ⇒ (iii). Because D is not a valuation domain, there exist a and b in D such that $aD \cap bD = aM$ (Proposition 4.1). Now by the *-property

$$\begin{aligned} a^2M^2 &= ((a) \cap (b))^2 = (a^2) \cap (ab) \cap (ab) \cap (b^2) \\ &= a((a) \cap (b)) \cap b((a) \cap (b)) \\ &= a^2M \cap baM = a(aM \cap bM) = a(aM) = a^2M \end{aligned}$$

and from this it follows that $M^2 = M$.

(iii) ⇒ (iv). It follows from Theorem 2.3 of Dobbs [7].

(iv) ⇒ (v). Follows from Theorem 12 of [6].

That (v) ⇒ (i) follows from Corollary 4.3 and finally (ii) ⇒ (vi) and (vi) ⇒ (iii) are obvious.

Thus in a PVD which is not a valuation domain the *-property is equivalent also to the Sylvester property of [6].

In section 3 we showed that in a quasi local PVMD the *-property is equivalent to the pre-Schreier property. (Note that a PVMD is a GCD-domain if and only if it is Schreier). The results of this section lead to examples of non-PVMD's in which the *-property is equivalent to the pre-Schreier property. Indeed we can go a step further and produce a simple and crystal clear example of a pre-Schreier domain, which is not Schreier.

EXAMPLE 4.5. Let $S = \{X^r \mid r, \text{ rational} \geq 0\}$. Form the algebra R[S] over the field R of reals. It is easy to see that R[S] is a Bezout domain (every finitely generated ideal is principal). Let $P = \{f \in R[S] \mid f$ has zero constant term$\}$. Clearly P is a prime ideal and clearly $T = R[S] - P$ is the set of elements with non zero constant terms.

It is easy to see that $R[S] = R + P$ and because it is a Bezout domain we conclude that $(R[S])_T = R + (P)_T = V$ is a valuation domain with maximal ideal $(P)_T$. It is well known that then $D = Q + (P)_T$, where Q is the set of rationals, is a PVD (Cf. e.g. [8] Proposition 4.9). Now because of the definition of S, $(P)_T^2 = (P)_T$. Thus by Theorem 4.4, $Q + (P)_T$ is a pre-Schreier domain. That $Q + (P)_T$ is not integrally closed in obvious.

REMARKS 4.6.

(1) Example 4.5 is not the first ever example of a strictly pre-Schreier domain (Cf. [13] and the references there), nor is it a novel one ; yet it can be put to a novel use. We use it to give a concrete example of the fact that if D is strictly pre-Schreier and if Y is an indeterminate over D then D[Y] is not pre-Schreier. For this let $D = Q + (P)_T$ and consider in D[Y] the product $(\sqrt{2}\, X^r Y + X^r)(\sqrt{2}\, X^r Y - X^r)$. Now

$$X^{2r} \mid (\sqrt{2}\, X^r Y + X^r)(\sqrt{2}\, X^r Y - X^r) = 2X^{2r}Y^2 - X^{2r} = X^{2r}(2Y^2 - 1)$$

but we cannot split $2r = p + q$ in any possible way to get $X^p \mid \sqrt{2}\, X^r Y + X^r$ and $X^q \mid \sqrt{2}\, X^r Y - X^r$. The obvious conclusion is that for no positive rational p is X^p primal in D[Y].

(2) It is well known that a lattice ordered group is torsion free and Fuchs [9] (p. 9 Example 4), points out that in a Riezs group there may be elements of finite order. Example 4.5 can be used to demonstrate this fact conveniently as follows.

Let $D = R + P_T$ as in Example 4.5. Then as D is a pre-Schreier domain its group of divisibility

$G(D) = \{(x/y)D \mid x,y \in D - \{0\}\}$ under multiplication defined by $(x/y)D \cdot (u/v)D = (xu/yv)D$ and order defined by $(x/y)D \leq (r/s)D$ if $(r/s)D \subseteq (x/y)D$, is a Riesz group. (Here D is the identity).

Now let $a = \sqrt{2}\,X$ and $b = X$. Then $((a/b)D)^2 = (2X^2/X^2)D = 2D = D$. That is $(a/b)D$ is an element of order 2.

In fact it is easy to see that G(D) in this case is a Riesz group in which for every integer n there is an element $(u/v)D$ in G(D) such that $((u/v)D)^n = D$.

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REFERENCES

[1] D.D. Anderson and D.F. Anderson, Generalized GCD-domains, Comment. Math. Univ. St. Pauli, 23 (1979), 213-221.

[2] D.D. Anderson, D.F. Anderson and E.W. Johnson, Some ideal theoric conditions on a noetherian ring, Houston J. Math. 7 (1981), 1-10.

[3] D.F. Anderson and D.E. Dobbs, On the product of ideals, Canad. Math. Bull. 26 (1983), 106-114.

[4] N. Bourbaki, Commutative Algebra, Addison-Wesley, Reading Mass., (1972).

[5] P.M. Cohn, Bezout rings and their subrings, Proc. Camb. Phil. Soc. 64 (1968), 251-264.

[6] W. Dicks and E.D. Sontag, Sylvester domains, J. Pure and Applied Alg. 13(3) (1978), 243-275.

[7] D.E. Dobbs, On the weak global dimension of pseudo-valuation domains, Canad. Math. Bull. 21(2) (1978), 159-164.

[8] D.E. Dobbs, Coherence, ascent of going down and pseudo-valuation domains, Houston J. Math. 4(4) (1978), 551-567.

[9] L. Fuchs, Riesz groups, Ann. Scoula Norm. Sup. Pisa 19 (1965), 1-34

[10] R. Gilmer, Multiplicative Ideal Theory, Marcel Dekker, New York (1972).

[11] M. Griffin, Some results on v-multiplication rings, Canad. J. Math. 19 (1967), 710-722.

[12] M. Griffin, Rings of Krull type, J. Riene Angew. Math. 229 (1968), 1-27.

[13] S. McAdam and D.E. Rush, Schreier rings, Bull. London Math. Soc. 10 (1978), 77-80.

[14] R.G. Swan, n-generator ideals in Prüfer domains, Pacific J. Math. 111 (1984), 433-446.

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