ON GENERALIZED DEDEKIND DOMAINS

MUHAMMAD ZAFRULLAH

Throughout this note the letters D and K denote a commutative integral domain with 1 and its field of fractions.

Let $F(D)$ denote the set of non-zero fractional ideals of D. For $A, B \in F(D)$ it is not generally true that $(AB)^{-1} = A^{-1}B^{-1}$. So it is natural to ask for a characterization of intergal domains satisfying,

(GD) *for all* $A, B \in F(D)$; $(AB)^{-1} = A^{-1}B^{-1}$.

We show that an integral domain D satisfies (GD), if, and only if, for each $A \in F(D)$ the ideal $A_v = (A^{-1})^{-1}$ is invertible. It is easy to see that a Dedekind domain satisfies (GD) and for this reason we call the integral domains satisfying (GD), *generalized Dedekind domains* (*G-Dedekind domains*).

The examples of G-Dedekind domains range from UFD's and locally factorial Krull domains to the ring of entire functions and rank one valuation domains with complete value groups. The G-Dedekind domains are similar to the Dedekind domains in that they are completely integrally closed and locally GCD-domains and different in that a quotient ring of a G-Dedekind domain may not be a G-Dedekind domain and a ring of polynomials over a G-Dedekind domain is again a G-Dedekind domain.

In studying the G-Dedekind domains we shall use the notion of *v-operation*. To ensure convenience and completeness we include some preparatory remarks about the v-operation and related notions.

Let $F(D)$ denote the set of non-zero fractional ideals of D. Associated to each $A \in F(D)$ is the fractional ideal $(A^{-1})^{-1} = A_v$. The map $A \mapsto A_v$ on $F(D)$ is a $*$-*operation* called the *v-operation. The reader may consult Sections* 32 *and* 34 *of* [4] *for the definition and properties of* $*$-*operations. For our purposes we note the following. Let* $A, B \in F(D)$ and let $x \in K - \{0\}$. Then

(1) $(xD)_v = xD$, $(xA)_v = xA_v$,
(2) $A \subseteq A_v$ *and if* $A \subseteq B$ *then* $A_v \subseteq B_v$,
(3) $(A_v)_v = A_v$,
(4) $(AB)_v = (A_vB)_v = (A_vB_v)_v$, we shall refer to these equations as defining *v-multiplication*,
(5) $A^{-1} = (A_v)^{-1} = (A^{-1})_v$.

An ideal $A \in F(D)$ is called a *v-ideal* if $A = A_v$, and a *v-ideal of finite type* if $A = B_v$ for some finitely generated $B \in F(D)$. An ideal A is called a *t-ideal* if $A = \bigcup F_v$, where F ranges over finitely generated D-submodules of A. A v-ideal is a t-ideal. An integral ideal maximal with respect to being a t-ideal is called a *maximal t-ideal*. A maximal t-ideal is a prime ideal (*cf* [9]; Cor. 1 to Th. 9, p. 30).

[MATHEMATIKA, 33 (1986), 285–295]

Obviously an invertible ideal is a v-ideal and hence a t-ideal. Further, $A \in F(D)$ is called *v-invertible* if there exists $B \in F(D)$ such that $(AB)_v = D$. It is well known that if, for all $A \in F(D)$, A is v-invertible then D is completely integrally closed ([4], Th. 34.3). Further, D is said to be a *Mori domain* if the set of integral v-ideals of D satisfies ascending chain condition. According to Querre ([11], Th. 1) an integral domain D is a Mori domain, if, and only if, for all $A \in F(D)$, there exists a finitely generated $B \in F(D)$ such that $B \subseteq A$ and $A_v = B_v$. It is well known that *a completely integrally closed integral domain is a Krull domain, if, and only if, it is Mori* (*cf* e.g. [4], Ex. 15, p. 556).

An integral domain D is called a *Prüfer v-multiplication domain* (*PVMD*) if the set $H(D)$ of v-ideals of finite type of D is a group under the v-multiplication described in (4) above. In particular, Krull domains are PVMD's. According to Griffin ([5], Th. 5) D is a PVMD, if, and only if, for every maximal t-ideal P of D, D_P is a valuation domain. An integral domain D is called a *$*$-domain* if, for all $x_1, \ldots, x_m; y_1, \ldots, y_n \in K - \{0\}$, we have

$$\left(\bigcap (x_i)\right)\left(\bigcap (y_j)\right) = \bigcap (x_i y_j).$$

We shall call D *locally* X to indicate the fact that for each maximal ideal M, D_M has the property X. It was shown in [17] that D is a $*$-domain, if, and only if, it is locally a $*$-domain. Also indicated in [17] was the fact that a GCD-domain is a $*$-domain. Hence a locally GCD-domain is a $*$-domain. According to [17] a PVMD is a locally GCD-domain, if, and only if, it is a $*$-domain (Cor. 3.4). So a Krull domain is locally factorial, if, and only if, it is a $*$-domain. Finally a prime ideal P of D minimal over an ideal of the type $(a):(b) \neq D$) is called an *associated prime* of D. According to ([10], Cor. 3.3) an integral domain D is a locally factorial Krull domain, if, and only if, every associated prime of D is invertible.

This note is split into two sections. In the first section we include the basic theory and some rather obvious examples and counter examples. We first prove that, for $A \in F(D)$, A_v is invertible, if, and only if, for all $B \in F(D)$, $(AB)^{-1} = A^{-1}B^{-1}$ (Lemma 1.2). Using this we can easily show that D is a G-Dedekind domain, if, and only if, for all $A \in F(D)$, A_v is invertible. We also indicate that a G-Dedekind domain is completely integrally closed (Cor. 1.4). Because every v-ideal in a G-Dedekind domain is invertible, every v-ideal of finite type is invertible and so a G-Dedekind domain is a *generalized GCD* (GGCD-) domain of [1]. This leads to the question, "What are the integral domains for which $(AB)^{-1} = A^{-1}B^{-1}$ for all finitely generated $A, B \in F(D)$?" The answer (Prop. 1.6) is that these are precisely the $*$-domains. We also show in this section that the ring of polynomials over a G-Dedekind domain is a G-Dedekind domain (Th. 1.9). We close this section with a list of equivalent conditions for an intergal domain to be a G-Dedekind Krull domain (Th. 1.10). Some of these conditions are: (1) D is Mori and a $*$-domain (2) every associated prime of D is invertible (3) every t-ideal of D is invertible and (4) D is a Krull domain such that the product of every two v-ideals is again a v-ideal.

In the second section we include some interesting examples and some questions. We show that the ring of entire functions is a G-Dedekind domain (Ex. 2.1), and use the fact that some quotient rings of the ring of entire functions are not completely integrally closed to deduce that a quotient ring of a G-Dedekind domain need not be a G-Dedekind domain (Cor. 2.3). We also show, using the fact that there is a one to one correspondence between the non-zero ideals of a valuation domain and the upper classes of the Dedekind cuts (*cf* [14], p. 10) that a rank one valuation domain with a complete group of divisibility has the property that, for all $A \in F(D)$, A_v is principal (Th. 2.6). So a valuation domain with the set of reals (under addition) as its group of divisibility is an example of a G-Dedekind domain. We also show that a valuation domain with the set Q of rational numbers (under +) as its group of divisibility is not a G-Dedekind domain (Ex. 2.7). For the questions the reader will have to turn to the last part of this section; as some of the questions need some motivation which cannot be provided here.

§1. *Basic theory and examples.* Having dealt with the introduction and motivation already, we start this section with the statement of the following theorem.

1.1. THEOREM. *An integral domain D is a G-Dedekind domain, if, and only if, for all $A \in F(D)$, A_v is invertible.*

Instead of proving this theorem as it is, we prove a more general result in the form of a lemma.

1.2. LEMMA. *Let $A \in F(D)$. A_v is invertible, if, and only if, $(AB)^{-1} = A^{-1}B^{-1}$, for all $B \in F(D)$.*

Proof. We note, to start with, that if, for all $B \in F(D)$, $(AB)^{-1} = A^{-1}B^{-1}$ then $(A_vB)^{-1} = A^{-1}B^{-1}$. This follows from the fact that $(AB)^{-1} = ((AB)_v)^{-1} = ((A_vB)_v)^{-1} = (A_vB)^{-1}$.

Now let $A \in F(D)$ and suppose that A_v is invertible. Then $A_vA^{-1} = D$. Let $B \in F(D)$ and let $x \in (AB)^{-1}$. Then $xAB \subseteq D$ and so $xA \subseteq B^{-1}$. Because B^{-1} is a v-ideal we have $xA_v \subseteq B^{-1}$. Now multiplying both sides by A^{-1} we get $x \in A^{-1}B^{-1}$. Now let $x \in A^{-1}B^{-1}$. Because A_v is invertible we get, on multiplying both sides by A_v, $xA_v \subseteq B^{-1}$ and so $xA_vB \subseteq D$ which gives $x \in (A_vB)^{-1} = (AB)^{-1}$. From these considerations it follows that, if A_v is invertible, then $(AB)^{-1} = A^{-1}B^{-1}$, for all $B \in F(D)$.

Conversely suppose that for all $B \in F(D)$, $(AB)^{-1} = A^{-1}B^{-1}$. Then $(A_vB)^{-1} = A^{-1}B^{-1}$ too. Now put $B = A^{-1}$. Then

$$(A_vA^{-1})^{-1} = (A_v)^{-1}(A^{-1})^{-1} = A^{-1}A_v = A_vA^{-1}.$$

Now as $A^{-1} = A_v^{-1}$ we have $A_vA^{-1} \subseteq D$ and so $(A_vA^{-1})^{-1} \supseteq D$. But then $(A_vA^{-1})^{-1} = A_vA^{-1}$ forces $A_vA^{-1} = D$, which means that A_v is invertible.

Recalling that an invertible ideal is a v-ideal we make the following statement.

1.3. COROLLARY. *An ideal $A \in F(D)$ is invertible, if, and only if, A is a v-ideal and, for all $B \in F(D)$, $(AB)^{-1} = A^{-1}B^{-1}$.*

After the rather straight-forward characterization of G-Dedekind domains we proceed to determine some of their multiplicative properties.

1.4. COROLLARY. *A G-Dedekind domain is completely integrally closed.*

Proof. Let D be G-Dedekind. Then, for all $A \in F(D)$, $(AA^{-1})_v = (A_vA^{-1})_v = (D)_v = D$, and so every non-zero ideal of D is v-invertible. This, as we have already indicated means that D is completely integrally closed.

The proof of the following corollary has been indicated in the introduction.

1.5. COROLLARY. *A G-Dedekind domain is a* GGCD-*domain.*

According to [1] a GGCD-domain is locally a GCD-domain. So a G-Dedekind domain is locally GCD and hence a $*$-domian. According to ([17], Cor. 3.4) D is a GGCD-domain, if, and only if, it is a PVMD and a $*$-domain. That a G-Dedekind domain is both a PVMD and a $*$-domain can also be derived from the definition. For the PVMD property we note that,, for every $A \in F(D)$, A^{-1} is finitely generated and so the same is true for finitely generated A. For the $*$-property on the other hand we state the following simple proposition.

1.6. PROPOSITION. *An integral domain D is a $*$-domain, if, and only if, for all finitely generated $A, B \in F(D)$, $(AB)^{-1} = A^{-1}B^{-1} = (A_vB_v)^{-1}$.*

Proof. Let $A = (a_1, \ldots, a_m)$ and $B = (b_1, \ldots, b_n)$ be in $F(D)$. Then $AB = (a_1b_1, \ldots, a_ib_j, \ldots, a_mb_n)$. Now if $A^{-1}B^{-1} = (AB)^{-1}$ we have

$$\left(\bigcap (1/a_i)\right)\left(\bigcap (1/b_j)\right) = \bigcap (1/(a_ib_j))$$

and selecting suitable A, B we can show that, for all sets $x_1, \ldots, x_m; y_1, \ldots, y_n \in K - \{0\}$, we have $(\bigcap (x_i))(\bigcap (y_j)) = \bigcap (x_iy_j)$.

Conversely if D is a $*$-domain then, for all $x_i, y_j \in K - \{0\}$, we have $(\bigcap (x_i))(\bigcap (y_j)) = \bigcap (x_iy_j)$. So, for A and B as above, we have $A^{-1}B^{-1} = (\bigcap (1/a_i))(\bigcap (1/b_j))$ and by the $*$-property this becomes $\bigcap (1/(a_ib_j) = (AB)^{-1}$.

The proof will be complete once we note that

$$(AB)^{-1} = ((AB)_v)^{-1} = ((A_vB_v)_v)^{-1} = (A_vB_v)^{-1}$$

for all $A, B \in F(D)$ and for every integral domain D.

The above proposition leads to an interesting characterization of GGCD-domains.

1.7. COROLLARY. *An integral domain D is a GGCD-domain, if, and only if, D is a $*$-domain and, for all finitely generated $A \in F(D)$, A^{-1} is of finite type.*

Proof. The "only if" part is obvious. For the "if" part all we have to show is that for all finitely generated $A \in F(D)$, A_v is invertible . . . given that D is a $*$-domain and A^{-1} is of finite type. So let A be finitely generated and let $A^{-1}=B_v$ where B is finitely generated. We note that $(A_vA^{-1})^{-1}=(AA^{-1})^{-1}=(AB)^{-1}$ and by the $*$-property (and Prop. 1.6) $(AB)^{-1}=A^{-1}B^{-1}=A^{-1}A_v$. This gives $(A_vA^{-1})^{-1}=A_vA^{-1}$ and hence, as in Lemma 1.2., we have $A_vA^{-1}=D$.

The above considerations lead to a rather interesting set of characterizations of G-Dedekind domains.

1.8. Proposition. *For D the following are equivalent.*

(1) *D is a G-Dedekind domain.*

(2) *D is completely integrally closed and for all $A, B\in F(D)$, $(AB)_v=A_vB_v$.*

(3) *For every ideal $A\in F(D)$, A_v is of finite type and D is a $*$-domain.*

Proof. (1)⇒(2). Because D is a G-Dedekind domain it is completely integrally closed (Cor. 1.4) and because for all $A, B\in F(D)$, $(AB)^{-1}=A^{-1}B^{-1}$ we have

$$(AB)_v=((AB)^{-1})^{-1}=(A^{-1}B^{-1})^{-1}=A_vB_v.$$

(2)⇒(1). Let $A\in F(D)$. Then as D is completely integrally closed $(AA^{-1})_v=D$ ([4], Th. 34.3). But by (2) $(AA^{-1})_v=A_v(A^{-1})_v=A_vA^{-1}$. So for each $A\in F(D)$, A_v is invertible and hence, by Th. 1.1, D is a G-Dedekind domain.

(1)⇒(3). A G-Dedekind domain is a $*$-domain and because, for every $A\in F(D)$, A_v is invertible it is of finite type.

(3)⇒(1). Note that $(AB)^{-1}=(A_vB_v)^{-1}$ for all $A, B\in F(D)$. By (3) each of A_v, B_v is of finite type and by (3) again (and Prop. 1.6) $(A_vB_v)^{-1}=A_v^{-1}B_v^{-1}=A^{-1}B^{-1}$. But $(A_vB_v)^{-1}=(AB)^{-1}$.

The first step towards examples of a class of integral domains is to see if the class is closed under quotient ring and polynomial ring formations. We shall give an example in the next section to show that if D is a G-Dedekind domain and S is a multiplicative set in D then it is not necessary that D_S should be a G-Dedekind domain. For polynomials we state the following result.

1.9. Theorem. *If D is a G-Dedekind domain and X is an indeterminate over D, then $D[X]$ is again a G-Dedekind domain.*

Proof. It is sufficient to show that every integral v-ideal of $D[X]$ is invertible. For this let A be an integral v-ideal of $D[X]$. Then, as D is integrally closed, according to Querre ([12]; Lemme 3.2), if $A\cap D\neq(0)$ we have $A\cap D$ a v-ideal of D such that $A=(A\cap D)[X]$, and if $A\cap D=(0)$ we have $A=fA_1[X]$ for some $f\in D[X]$ and for some v-ideal $A_1\in F(D)$.

Now it is easy to see that if F is an invertible ideal in $F(D)$ then $F[X]$ is invertible. So, using the fact that every v-ideal of D is invertible we can conclude that $A=(A\cap D)[X]$ (for $A\cap D\neq(0)$) and $A=fA_1[X]$ (for $A\cap D=(0)$) are both invertible.

Theorem 1.9, not only acts as an example schema but also it acts to differentiate G-Dedekind domains from Dedekind domains. For if D is a Dedekind domain then $D[X]$ is not Dedekind but it is G-Dedekind.

A G-Dedekind domain which is Krull, Mori or Noetherian becomes a *locally factorial Krull domain*, generally known as a π-*domain* (*cf* [4] and [3]). In the following we list a set of equivalent conditions under which an integral domain becomes a π-domain.

1.10. THEOREM. *The following are equivalent for an integral domain D.*

(1) *D is a G-Dedekind Krull domain.*

(2) *D is G-Dedekind and Mori.*

(3) *D is Krull and locally factorial.*

(4) *D is Krull and a* $*$*-domain.*

(5) *D is Krull and, for all* $a, b, c, d \in D-\{0\}$,

$$((a)\cap(b))((c)\cap(d))=(ac)\cap(ad)\cap(bc)\cap(bd).$$

(6) *D is Mori and locally factorial.*

(7) *D is Mori and a* $*$*-domain.*

(8) *D is Mori and GGCD.*

(9) *every t-ideal of D is invertible.*

(10) *every associated prime of D is invertible.*

(11) *D is Krull such that the product of any two v-ideals is again a v-ideal.*

(12) *D is G-Dedekind, every quotient ring of D is G-Dedekind and every rank one prime ideal of D is invertible.*

Proof. We shall prove the equivalence of (1)–(11) using the usual cyclic scheme. Then we show that (1, 4, 10)$\Rightarrow$(12) and (12)$\Rightarrow$(10).

(1)$\Rightarrow$(2). Obvious because Krull is Mori.

(2)$\Rightarrow$(3). Because a G-Dedekind domain is completely integrally closed and because a completely integrally closed Mori domain is Krull we conclude that D is Krull. Now being G-Dedekind D is GGCD and hence locally GCD (*cf* [1]). Being Krull and locally GCD makes it locally factorial.

(3)$\Rightarrow$(4). Obvious because a locally GCD-domain is a $*$-domain.

(4)$\Rightarrow$(5). This is obvious too.

(5)$\Rightarrow$(6). According to ([2], Th. 3.8) (5) implies that D is locally factorial and locally factorial Krull implies locally factorial Mori.

(6)$\Rightarrow$(7). The property of being locally GCD implies the $*$-property.

(7)$\Rightarrow$(8). Because D is Mori, for each $A \in F(D)$, A_v is of finite type. Now, by Proposition 1.8, D is G-Dedekind and hence GGCD.

(8)$\Rightarrow$(9). We note that in a Mori domain every t-ideal is a v-ideal. This follows from the fact that in a Mori domain for every $A \in F(D)$ there is a finitely generated $B \subseteq A$ such that $A_v = B_v$. Now $A_v \supseteq A_t = \bigcup F_v$ where F ranges over finitely generated D-submodules of A. But then B is a finitely generated D-submodule of A. So $A_v \supseteq A_t = \bigcup F_v \supseteq B_v = A_v$ which gives $A_t = A_v$ for all $A \in F(D)$. Now by the GGCD property every v-ideal of finite type is invertible and by the Mori property every t-ideal is a v-ideal of finite type. So, for all $A \in F(D)$, A_t is invertible.

(9)$\Rightarrow$(10). This follows directly because every associated prime is a t-ideal (*cf* [16], p. 1703).

(10)$\Rightarrow$(11). According to ([10]. Cor. 3.3), (10) holds, if, and only if, D is a locally factorial Krull domain; that is, if, and only if, D is a $*$ and a Krull domain. But in a Krull domain, for every $A \in F(D)$, A_v is of finite type while the $*$-property implies $(A_vB_v)^{-1} = A_v^{-1}B_v^{-1}$ or $(AB)^{-1} = A^{-1}B^{-1}$ for all $A, B \in F(D)$.

(11)$\Rightarrow$(1). Being Krull, D is completely integrally closed and now part (2) of Proposition 1.8 applies.

(1, 4, 10)$\Rightarrow$(12). Suppose that D is G-Dedekind and Krull. Then D is a Krull $*$-domain (4). Because every quotient ring of a Krull and a $*$-domain is Krull and $*$, every quotient ring of D is G-Dedekind and by (10) every rank one prime ideal is invertible.

(12)$\Rightarrow$(10). Since D is G-Dedekind it is a PVMD and so for every maximal t-ideal P of D, D_P is a valuation domain. Further, since every quotient ring is G-Dedekind the valuation ring D_P must be of rank one. So every maximal t-ideal of D is of rank one. Consequently every associated prime, being a t-ideal and hence being contained in a maximal t-ideal, must be of rank one. So every rank one prime is invertible implies that every associated prime is invertible and this is exactly (10).

§2. *Some interesting examples, counter examples and questions.*

2.1. *Example.* The ring R of entire functions is a G-Dedekind domain.

Illustration. Let $f(z)$ be an entire function and let Z_f denote the *algebraic* set of zeros of f. The set Z_f is algebraic in the sense that if f has n zeros at z then z appears in Z_f, n times. It is well known that if Z is an algebraic set then there exists an entire function f such that $Z = Z_f$. We note that an entire function which has no zeros is a unit and so we take $\varnothing = Z_1$.

Let A be an ideal of R. Then according to Henriksen [8], A is *fixed* if $Z = \bigcap Z_f \neq \varnothing$ (where f ranges over A) and *free* if $Z = \bigcap Z_f = \varnothing$ (f ranges over A). If $\bigcap_{f \in A} Z_f \neq \varnothing$ then there exists $g \in R$ such that $\bigcap_{f \in A} Z_f = Z_g$ and obviously $g \mid f$ for all $f \in A$. According to Helmer ([7], Th. 7) $g = \mathrm{GCD}(f \mid f \in A)$. If on the other hand $\bigcap_{f \in A} Z_f = \varnothing$ we conclude that there is no non-zero non-unit common factor of members of A and so $1 = \mathrm{GCD}(f \mid f \in A)$; in this case.

Now because R is a Bezout ring (every finitely generated ideal is principal (Helmer [7], Th. 9)) and because every fractional ideal A can be written as $A = B/x$ where B is integral, to show that R is G-Dedekind it is sufficient to show that, for every integral ideal B of R, B_v is principal. To do this we prove a slightly more general result; noting that a Bezout domain is GCD too.

2.2. Lemma. *Let D be a GCD-domain and let A be a non-zero integral ideal of D. Then $A_v = \bigcap_{A \subseteq xD} xD$.*

Proof. By definition $A_v = \bigcap_{A \subseteq (x/y)D} (x/y)D$, where $x, y \in D - \{0\}$. Now because D is a GCD-domain we can assume each pair x, y coprime. But then $A \subseteq (x/y)D$ implies $yA \subseteq xD$ which in turn implies that $A \subseteq xD$ and indeed as $xD \subseteq (x/y)D$ we can discard $(x/y)D$ and conclude that $A_v = \bigcap_{A \subseteq xD} xD$.

So if $g = \mathrm{GCD}(A)$ then, for all $x \in R - \{0\}$ with $A \subseteq xR$, we have $x \mid g$ and so $xR \supseteq gR$. Whence it follows that $A_v = \bigcap_{A \subseteq xR} xR = gR$. Further if $\mathrm{GCD}(A) = 1$ then $A_v = \bigcap_{A \subseteq xR} xR$ implies that $A_v = R$. This completes the illustration.

In [15] an exercise from ([4], Ex. 20, p. 432) was brought to life and it was shown that, if A is a finitely generated ideal of D, then $(AD_S)_v = (A_vD_S)_v$. Using this result it can be shown easily that if A is a v-ideal of finite type, that is, if $A = B_v$, where B is finitely generated, then $(AD_S)_v$ is a v-ideal of finite type. For $AD_S = B_vD_S$ and $(AD_S)_v = (B_vD_S)_v = (BD_S)_v$. It is natural to ask if there is an example of a fractional ideal A such that $(AD_S)_v \neq (A_vD_S)_v$. The answer is that the ring of entire functions affords infinitely many such examples. To take as one example let P be a non-maximal prime ideal of R of rank greater than one and let M be a maximal ideal containing P. Because R is Bezout it can be shown that R_M is a valuation domain and PR_M is a non-maximal prime of R_M. But then PR_M is the intersection of all the principal ideals containing it and hence is a v-ideal and obviously PR_M is not principal. But P_v is principal and so is $(P_vR_M)_v$. Now a principal ideal cannot be equal to a non-principal ideal.

To draw another interesting indirect conclusion from Example 2.1, we recall that a G-Dedekind domain is completely integrally closed. So if an integral domain is not completely integrally closed it cannot be G-Dedekind. Consequently, *no valuation domain of rank greater than one can be a G-Dedekind domain.* Now the ring R of entire functions does have prime ideals P which are of rank greater than one. And because R is a Bezout domain, R_P is a valuation domain of rank greater than one. This leads to the now obvious conclusion.

2.3. COROLLARY. *If D is a G-Dedekind domain it is not necessary that a ring of quotients of D should also be a G-Dedekind domain.*

Now we prepare to give an example of a G-Dedekind domain which is a non-discrete rank one valuation domain.

Let G be a totally ordered group. A subset H of G^+ is called an *upper class of positive elements*, if, for all $h \in H$, $k > h$ implies $k \in H$. Let $\mathbf{v}$ be a valuation on a field K, let V be its ring and let $G(V)$ be it group of divisibility (or value group). It is well-known that there is a one–one correspondence between $I(V)$ the set of non-zero integral ideals of V and $C(G(V))$ the set of upper classes of positive elements of $G(V)$ (*cf.* Schilling [14], p. 10). This correspondence is given by $A \mapsto \mathbf{v}(A) = \{\mathbf{v}(a) \mid a \in A\}$. If $\mathbf{v}$ is of rank one, $G(V)$ can be regarded as a subgroup of the group of real numbers under addition (Schilling [14], Th. 1, p. 6). A lattice ordered group G is said to be *complete*, if, for every subset S (of G) which is bounded from *above* (*below*), $\sup(S)$ ($\inf(S)$) belongs to G. Now $G(V)$ being totally ordered is lattice ordered and so its being complete or non-complete can be considered. Obviously if $\mathbf{v}$ is of rank one for every integral ideal A, $\mathbf{v}(A)$ is bounded from below and so has an infimum in $\mathbb{R}$ the set of reals and this $\inf(\mathbf{v}(A))$ may or may not belong to $G(V)$. If however $G(V)$ is complete $\inf(\mathbf{v}(A))$ belongs to $G(V)$ for every non-zero ideal A.

2.4. LEMMA. *Let V be a rank one valuation domain and let A be a non-zero integral ideal of V. If* $\inf(v(A))=0$ *then either $A=V$ or $A=M$ the maximal ideal of V.*

Proof. Clearly, for all $a\in A$, $v(a)\geqslant 0$ and if $0=\inf(v(A))\in v(A)$ we have $1\in A$ and so $A=V$. If on the other hand $0=\inf(v(A))\notin v(A)$ then, for all $x\in M$, there exists $y\in A$ such that $y|x$. For if not and if, for some $x\in M$, there exists no $y\in A$ such that $y|x$ then, for all $a\in A$, $v(a)>v(x)>0$ and this contradicts $\inf(v(A))=0$. So, for all $x\in M$, there is $y\in A$ such that $y|x$. But this means that $M\subseteq A$.

2.5. LEMMA. *Let V be a rank one valuation domain with a complete value group $G(V)$. Then, for every non-maximal non-zero integral ideal A of V, $A=xM$ or $A=xV$.*

Proof. If A is a proper ideal of V then, for all $a\in A$, $v(a)>0$. Now, by the completeness of $G(V)$ and by Lemma 2.3., there exists $r\in(G(V))^+$ such that $0<r=\inf(v(A))$. So there exists $x\in M$ such that $v(x)=r$ and $A\subseteq xV$, and there is no ideal between A and xV. (For if there were an ideal between A and xV then $r\neq\inf(v(A))$.) Now $A/x\subseteq V$ and $\inf(v(A/x))=0$. But then, by Lemma 2.3, $A/x=V$ or $A/x=M$.

2.6. THEOREM. *Let V be a rank-one valuation domain with a complete value group $G(V)$. Then, for every integral non-zero non-maximal ideal A of V, A_v is principal and hence V is a G-Dedekind domain.*

Proof. There are two cases: V is discrete or V is non-discrete. If V is discrete every ideal of V is principal and we have nothing to prove. If V is non-discrete then the maximal ideal M of V is such that $M^{-1}=V$ and so $M_v=V$. Now if A is non-maximal then, by Lemma 2.5, $A=xV$ *or* $A=xM$ and this gives $A_v=xV$.

It is easy to see that in a valuation domain (in fact in any Prüfer domain) every non-zero ideal is a t-ideal... because every finitely generated ideal is principal (invertible) and hence a v-ideal. If the maximal ideal is non-principal it is a good example of a t-ideal which is not a v-ideal. But if the set of real numbers is the value group of V we can give elementary examples of t-ideals A such that A_v is principal and $A_v\neq A$. One simple example could be, for any real $r>0$, the ideal $A=\{a\in V|v(a)>r\}$, and it is easy to see that $A_v=bV$ where b is such that $v(b)=r$.

This discussion will be incomplete if we did not give an example of a non-principal v-ideal in a rank one valuation domain.

2.7. *Example.* Let V be a valuation domain with value group $G(V)$ equal to Q the set of rationals under addition. Then (a) V has integral ideals A with A_v principal and (b) V has integral ideals B with B_v non-principal and $B_v=B$.

Illustration. For (a) we have $A=\{x\in V|v(x)>2\}$. Because $\inf(v(A))=2\notin v(A)$, A is non-principal. Further, because $2\in G(V)$ there exists $x\in V$ such that $v(x)=2$ and so $A \subseteq xV$ and there is no ideal between A and xV. Consequently $A/x \subseteq V$ and $\inf(\mathrm{v}(A/x))=0$, which gives $A=xM$ and so $A_v=xV$.

For (b) let $B=\{x \in V \mid (\mathrm{v}(x))^2>2\}$. Then $\inf(\mathrm{v}(B)) \in \mathbb{R}-\mathbb{Q}$. Since V is a valuation domain all the principal ideals which are not contained in B contain B. So if $x \notin B$ then $B \subseteq xV$ and $B_v \subseteq xV$ that is, if there is $y \in B_v - B$, then $B_v = yV$. Clearly $\mathrm{v}(y)^2<2$. For if $\mathrm{v}(y)^2>2$ then $y \in B$ and $\mathrm{v}(y)^2=2$ is not possible. But as $\mathbb{Q}$ is dense in $\mathbb{R}$, there exists $r \in \mathbb{Q}$ such that $\mathrm{v}(y)<r<\sqrt{2}$ and, because $\mathbb{Q}$ is the value group of V, we have $a \in V$ such that $\mathrm{v}(a)=r$. But then $\mathrm{v}(a)>\mathrm{v}(y)$ implies that $\mathrm{v}(a) \in \mathrm{v}(B_v)$ and that $a \in B_v$. Further because $\mathrm{v}(a)<\sqrt{2}$ we have $a \in B_v - B$ and this means that $B_v = aV$ also. This leads to $B_v = yV = aV$ where $\mathrm{v}(y) \neq \mathrm{v}(a)$ which is an obvious contradiction. From this we conclude that $B_v = B$.

Turning to the problems and questions we note that a quotient ring of a G-Dedekind domain is not necessarily a G-Dedekind domain. So we do have the following problem.

2.8. *Problem.* Characterize G-Dedekind domains whose quotient rings are also G-Dedekind domains.

Obviously in such a G-Dedekind domain D every prime t-ideal P is of rank one and D_P is a rank one valuation domain with a complete value group.

To state the next problem we need to prepare a bit. An integral domain D is called a *ring of Krull type* [6] *if it has a family* $\mathbb{F}=\{P_i\}_{i \in I}$ *of prime ideals such that*:

(1) $D=\bigcap D_{P_i}$;

(2) *for each* $i \in I$, D_{P_i} *is a valuation domain*; *and*

(3) *each non-zero non-unit of* D *belongs to only a finite number of* P_i.

A ring of Krull type is called a *generalized Krull domain* (*GKD*) *if its defining family of prime ideals consists of rank one primes alone.* GKD's were studied by Ribenboim [13]. According to Proposition 21 of [6] a completely integrally closed ring of Krull type is a GKD. Further it is easy to show that, if D is a GKD, then so is $D[X]$, where X is an indeterminate over D. Now the existence of a non-discrete valuation domain with a complete value group establishes the existence of G-Dedekind domains which are GKD's and not Krull. Moreover the fact that if D is G-Dedekind then so is $D[X]$ ensures that there is a plenty of such examples.

2.9. *Problem.* Characterize G-Dedekind GKD's.

2.10. *Question.* Is it true that the G-Dedekind domains of Problem 2.8 are GKD's? (The answer seems to be no, but this author cannot think of an example at present.)

2.11. *Question.* Let p be a prime element in D such that $\bigcap(p^n)=(0)$ and let $S=\{p^n\}_0^\infty$. If D_S is a G-Dedekind domain (or a GGCD- or a $*$-domain) is D a G-Dedekind (or a GGCD- or a $*$-) domain?

If the answer to any case of Question 2.11 is in the affirmative and simple it will provide a simple proof of the fact that a regular local ring is a UFD.

Acknowledgements. This research was carried out during the 1985–86 session, while this author worked at the Department of Mathematics, University College, London, as an honorary research fellow. The author is grateful to the department and to the college for providing the much needed year-long sanctuary.

References

1. D. D. Anderson and D. F. Anderson. Generalized GCD-domains. *Comment. Math. Univ. St. Pauli*, 23 (1979), 213–221.
2. D. D. Anderson, D. F. Anderson and E. W. Johnson. Some ideal theoretic conditions on a Noetherian ring. *Houston J. Math.*, 7 (1981), 1–10.
3. D. D. Anderson and J. Matijevic. Graded π-rings. *Can. J. Math.*, 31 (3) (1979), 449–457.
4. R. Gilmer. *Multiplicative Ideal Theory* (Marcel Dekker, New York, 1972).
5. M. Griffin. Some results on v-multiplication rings. *Can. J. Math.*, 19 (1967), 710–722.
6. M. Griffin. Rings of Krull type. *J. Reine Angew. Math.*, 229, (1968), 1–27.
7. O. Helmer. Divisibility properties of integral functions. *Duke Math. J.*, 6 (1940), 345–356.
8. M. Henriksen. On the ideal structure of the ring of entire functions. *Pacific J. Math.*, 2 (1952), 179–184.
9. P. Jaffard. *Les Systemes d'Ideaux* (Dunod, Paris, 1960).
10. S. Malik, J. Mott and M. Zafrullah. On t-invertibility (*preprint*).
11. J. Querre. Sur une propriete des anneaux de Krull. *Bull. Sc. Math.* 2° serie 95 (1971), 341–354.
12. J. Querre. Ideaux divisoriels d'un anneaux de polynomes. *J. Alg.*, 64 (1980), 270–284.
13. P. Ribenboim. Anneaux normaux reels a caractere fini. *Summa Brasil. Math.*, 3 (1956), 213–253.
14. O. Schilling. *The Theory of Valuations, Math. Surveys No. IV* (Amer. Math. Soc., 1950).
15. M. Zafrullah. On finite conductor domains. *Manuscripta Math.*, 24 (1978), 191–203.
16. M. Zafrullah. The v-operation and intersections of quotient rings of integral domains. *Comm. Algebra*, 13 (8) (1985), 1699–1712.
17. M. Zafrullah. On a property of pre-Schreier domains. To appear in *Comm. Algebra.*

Dr. M. Zafrullah,
c/o Department of Mathematics,
University College London,
Gower Street, London. WC1E 6BT

13A15: *COMMUTATIVE RINGS AND ALGEBRAS; General commutative ring theory; ideals.*
Received on the 22nd of August, 1986.