CHARACTERIZING DOMAINS OF FINITE $*$-CHARACTER

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To the memory of Professor Abdus Salam, a Nobel laureate Physicist and an Applied Mathematician with a pure heart. His services benefit friend and foe alike even after his death.

Abstract. For $*$ a star operation of finite type call a domain D a domain of finite $*$-character if every nonzero nonunit of D is contained in at most a finite number of maximal $*$-ideals. We prove a result that characterizes domains of finite $*$-character and outline its applications. Applications include characterization of Prufer and Noetherian domains of finite character and of domains of finite t-character.

An integral domain D is of finite character if every nonzero nonunit of D is contained in only a finite number of maximal ideals of D. The aim of this article is to prove results such as: a domain D is of finite character if and only if every nonzero finitely generated ideal of D is contained in at most a finite number of mutually comaximal finitely generated ideals. Consequently a Prüfer domain D is of finite character if and only if every invertible ideal of D is contained in at most a finite number of mutually comaximal invertible ideals, a result indicated somewhat laboriously in [16] and in earlier papers dealing with Bazzoni's Conjecture, cited in [16]. As another direct consequence we have the following result: A Noetherian domain D is of finite character if and only if every proper nonzero ideal of D is contained in at most a finite number of mutually comaximal proper ideals. We also recover most of the applications in [16]. Our approach involves the use of star operations, for which a basic introduction is provided below. Assuming familiarity with the star operations we aim to prove, and discuss some applications of, the following theorem.

Theorem 1. *Let D be an integral domain, $*$ a finite character star operation on D and let Γ be a set of proper, nonzero, $*$-ideals of finite type of D such that every proper nonzero $*$-finite $*$-ideal of D is contained in some member of Γ. Let I be a nonzero finitely generated ideal of D with $I^* \neq D$. Then I is contained in an infinite number of maximal $*$-ideals if and only if there exists an infinite family of mutually $*$-comaximal ideals in Γ containing I. Equivalently, with the same assumption on I, I is contained in at most a finite number of $*$-maximal ideals if and only if I is contained in at most a finite number mutually $*$-comaximal members of Γ.*

This Theorem is a, sort of, theorem schema where Γ is given various descriptions with varying values of $*$ and varying properties of D to fit the picture. For instance for $*$ the identity operation and for Γ as the set of all nonzero finitely generated ideals we get the results in the introduction. It may be instructive for a reader unfamiliar with star operations to assume the above "value" of Γ, disregard the star operation, read "$*$-finite $*$-ideal" as "finitely generated ideal" in the statement and proof of Theorem 1, along with its auxiliary lemma, and check.

For a more detailed study of star operations the reader may consult sections 32 and 34 of Gilmer's book [10] or [15]. For our purposes we include the following. Let D denote an integral domain with quotient field K and let $F(D)$ be the set of nonzero fractional ideals of D. A star operation $*$ on D is a function $*\colon F(D) \to F(D)$ such that for all $A, B \in F(D)$ and for all $0 \neq x \in K$

(a) $(x)^* = (x)$ and $(xA)^* = xA^*$,

(b) $A \subseteq A^*$ and $A^* \subseteq B^*$ whenever $A \subseteq B$,

(c) $(A^*)^* = A^*$.

For $A, B \in F(D)$ we define $*$-*multiplication* by $(AB)^* = (A^*B)^* = (A^*B^*)^*$ and $*$-*addition* by $(A+B)^* = (A^*+B)^* = (A^*+B^*)^*$. A fractional ideal $A \in F(D)$ is called a $*$-*ideal* if $A = A^*$ and a $*$-ideal of *finite type* if $A = B^*$ where B is a finitely generated fractional ideal. Also, $A \in F(D)$ is called $*$-finite if A^* is of finite type. A star operation $*$ is said to be of *finite character* if $A^* = \bigcup\{B^* \mid 0 \neq B$ is a finitely generated subideal of $A\}$. For $A \in F(D)$ define $A^{-1} = \{x \in K \mid xA \subseteq D\}$ and call $A \in F(D)$ $*$-*invertible* if $(AA^{-1})^* = D$. Clearly every invertible ideal is $*$-invertible for every star operation $*$. If $*$ is of finite character and A is $*$-invertible, then A^* is of finite type. The most well known examples of star operations are: the v-operation defined by $A \mapsto A_v = (A^{-1})^{-1}$, the t-operation defined by $A \mapsto A_t = \bigcup\{B_v \mid 0 \neq B$ is a finitely generated subideal of $A\}$, and the identity operation d that takes $A \mapsto A$ which is obviously of finite character. Given two star operations $*_1, *_2$ we say that $*_1 \leq *_2$ if $A^{*_1} \subseteq A^{*_2}$ for all $A \in F(D)$. Note that $*_1 \leq *_2$ if and only if $(A^{*_1})^{*_2} = (A^{*_2})^{*_1} = A^{*_2}$. By definition t is of finite character, $t \leq v$ while $\rho \leq t$ for every star operation ρ of finite character. If $*$ is a star operation of finite character then using Zorn's Lemma we can show that a proper integral ideal maximal w.r.t. being a $*$-ideal is a prime ideal, called a maximal $*$-ideal, and that every proper integral $*$-ideal is contained in a maximal $*$-ideal. We call proper ideals A, B of D $*$-comaximal if $(A+B)^* = D$, if $*$ is of finite character then $(A+B)^* = D$ means that A and B share no maximal $*$-ideals. Let us denote the set of all maximal $*$-ideals by $*-max(D)$. It can also be easily established that for a star operation $*$ of finite character on D we have $D = \bigcap_{M \in *-\max(D)} D_M$, [11]. For a domain D the function $A \mapsto A_w = \bigcap_{M \in t-\max(D)} AD_M$ is also a star operation of finite character and so $(A_w)_t = A_t$. An integral domain D is said to be of finite $*$-character, for a finite character star operation $*$, if every nonzero nonunit of D is contained in at most a finite number of maximal $*$-ideals of D. A t-finite ideal A is t-invertible if and only if A is t-locally principal i.e. for every $M \in t-\max(D)$ we have AD_M principal [12, Corollary 2.7]. An integral domain D is called a Prüfer v-multiplication domain (PVMD) if every nonzero finitely generated ideal of D is t-invertible. Griffin [11] called a PVMD of finite t-character a ring of Krull type. Call an integral domain D a t-Schreier domain if whenever A, B_1, B_2 are t-invertible ideals of D and $A \supseteq B_1B_2$, then $A = (A_1A_2)_t$ for some (t-invertible) ideals A_1, A_2 of D with $(A_i)_t \supseteq B_i$ for $i = 1, 2$. The t-Schreier domains were introduced in [9, page 380] as t-quasi-Schreier and studied in [8], where it was shown that if A is an ideal such that A_t is of finite type and $A_t \neq D$ then A is contained in a proper t-invertible t-ideal of D. Call D a $*$-sub-Prüfer domain, for a finite character star operation $*$, if every proper $*$-ideal of finite type of D is contained in a proper $*$-invertible $*$-ideal of D. Clearly as for a $*$-sub-Prüfer domain the set Γ consists of $*$-invertible $*$-ideals of D, Theorem 1 applies to $*$-sub-Prüfer domains. Clearly every Prüfer domain is a d-sub-Prüfer domain, with Γ consisting of proper invertible ideals, and every PVMD is a t-sub-Prüfer domain and so is a t-Schreier domain, both with Γ consisting of proper t-invertible t-ideals. So, Theorem 1 applies to all these domains and can be used to determine the finite character of these domains. In what follows we shall prove Theorem 1 and provide its applications. Towards the end of the paper we introduce the readers to a general approach which we hope will be of use in some other areas. Any unexplained terms are standard as in [10].

Call a proper $*$-finite $*$-ideal A of D *homogeneous* if A is contained in a unique maximal $*$-ideal.

Lemma 2. *Let D be a domain, $*$ a finite character star operation on D and let Γ be a set of $*$-finite $*$-ideals of D as described in Theorem 1. A proper $*$-finite $*$-ideal A of D is homogeneous if and only if whenever $B, C \in \Gamma$ are containing A, we get $(B, C)^* \neq D$.*

Proof. ($\Rightarrow$). Suppose that M is the only maximal $*$-ideal containing A and $B, C \in \Gamma$ ideals containing A. Then $B, C \subseteq M$, so $(B, C)^* \neq D$. ($\Leftarrow$). Suppose that A is contained in two distinct maximal $*$-ideals M_1, M_2. Hence $(M_1, M_2)^* = D$, so we can choose finitely generated ideals $F_i \subseteq M_i$, $i = 1, 2$, such that $A \subseteq F_i^*$ and $(F_1, F_2)^* = D$. There exist $G_1, G_2 \in \Gamma$ such that $F_i \subseteq G_i$, $i = 1, 2$. Hence $A \subseteq G_1, G_2$ and $(G_1, G_2)^* = D$. □

Proof. (of Theorem *1.)* The implication ($\Leftarrow$) is clear since a maximal $*$-ideal cannot contain two $*$-comaximal $*$-ideals. ($\Rightarrow$). Deny. So the following condition holds: ($\sharp$) there is no infinite family of mutually $*$-comaximal ideals in Γ containing I, Γ as defined in Theorem 1. First we show the following property: ($\sharp\sharp$) every proper $*$-finite $*$-ideal $I' \supseteq I$ is contained in some homogeneous ideal. Deny. As I' is not homogeneous, there exist $P_1, N_1 \in \Gamma$ such that $I' \subseteq P_1, N_1$ and $(P_1, N_1)^* = D$ (cf. Lemma 2). Since N_1 is not homogeneous, there exist $P_2, N_2 \in \Gamma$ such that $N_1 \subseteq P_2, N_2$ and $(P_2, N_2)^* = D$. Note that $(P_1, P_2)^* = (P_1, N_2)^* = D$. By induction, we can construct an infinite sequence $(P_k)_{k \geq 1}$ of mutually $*$-comaximal ideals in Γ with $I' \subseteq P_k$, $k \geq 1$. This fact contradicts condition ($\sharp$). So ($\sharp\sharp$) holds. To show that I is contained in at most a finite number of maximal $*$-ideals we proceed as follows. Let $\mathcal{S}$ be the family of sets of mutually $*$-comaximal members of Γ containing I. Then $\mathcal{S}$ is nonempty by ($\sharp\sharp$). Obviously $\mathcal{S}$ is partially ordered under inclusion. Let $A_{n_1} \subset A_{n_2} \subset ... \subset A_{n_r} \subset ...$ be an ascending chain of sets in $\mathcal{S}$. Consider $T = \cup A_{n_r}$. We claim that the members of T are mutually $*$-comaximal. For take $x, y \in T$, then $x, y \in A_{n_i}$, for some i, and hence are $*$-comaximal. Having established this we note that by ($\sharp$), T must be finite and hence must be equal to one of the A_{n_j}. Thus by Zorn's Lemma, $\mathcal{S}$ must have a maximal element $U = \{V_1, V_2, ..., V_n\}$. That each of V_i is homogeneous follows from the observation that if any of the V_i, say V_n by a relabeling, is nonhomogeneous then by Lemma 2 V_n is contained in at least two $*$-comaximal elements which by dint of containing V_n are $*$-comaximal with $V_1, ..., V_{n-1}$. This contradicts the maximality of U. Next let M_i be the maximal $*$-ideal containing V_i for each i and M be a maximal $*$-ideal that contains I and suppose that M does not contain any one of V_i. Then M is $*$-comaximal with each of the M_i. But then there is $x \in M \backslash \cup M_i$. Clearly (x, V_i) is contained in no maximal $*$-ideals and so $(x, V_i)^* = D$. But then $(I, x) \subseteq M$ is $*$-comaximal with each of V_i and by ($\sharp\sharp$), (I, x) is contained in a homogeneous $*$-ideal of finite type which being $*$-comaximal with V_i again contradicts the maximality of U. Consequently I is contained exactly in $M_1, M_2, ..., M_n$. The Equivalently part does not need extra proof being a contrapositive of the result that we have just proven. □

Remark 3. *(i) We have adopted this approach of stating results with their equivalents to (a) show how much ground can be covered and (b) to indicate the link of results proved in* [16]*, as we shall presently see. (ii) Theorem 1 can also be proved with Γ a set of $*$-ideals, but as the more general theorem would require more elaborate qualifying statements in the applications we have resisted the more general statement.*

Setting Γ as the set of all proper $*$-ideals of finite type we have the following corollary.

Corollary 4. *Let D be a domain, $*$ be a finite character star operation on D and I a nonzero finitely generated ideal of D with $I^* \neq D$. Then I is contained in an infinite number of maximal $*$-ideals if and only if there exists an infinite family of mutually $*$-comaximal proper $*$-finite $*$-ideals containing I. Equivalently, with the same assumptions on I, I is contained in at most a finite number of maximal $*$-ideals if and only if I is contained in at most a finite number of mutually $*$-comaximal $*$-ideals of finite type.*

Note that with $*$ and I as in Corollary 4, I being contained in an infinite family $\{F_\alpha^*\}$ of mutually $*$-comaximal proper $*$-finite $*$-ideals means that $\cap F_\alpha^* \neq (0)$ which in turn means that there is a proper finitely generated nonzero (preferably principal) ideal $J \subseteq F_\alpha^*$, for each α. This leads to the following statement.

Corollary 5. *Let D be a domain and let $*$ be a star operation of finite character. Then the following are equivalent: (1) There is an infinite family $\{F_\alpha^*\}$ of mutually $*$-comaximal proper $*$-finite $*$-ideals such that $\cap F_\alpha^* \neq (0)$ (2) There is a proper nonzero finitely generated ideal I with $I^* \neq D$ such that I is contained in an infinite family $\{F_\alpha^*\}$ of mutually $*$-comaximal proper $*$-finite $*$-ideals (3) There is a nonzero nonunit element $x \in D$ such that x belongs to an infinite number of maximal $*$-ideals. Equivalently every nonzero nonunit of D belongs to at most a finite number of maximal $*$ ideals if and only if every infinite intersection of mutually $*$-comaximal $*$-finite proper $*$ ideals is* 0.

Proof. (1) $\Leftrightarrow$ (2)$\Rightarrow$ (3) is obvious in view of the remarks prior to Corollary 5. For (3) $\Rightarrow$ (1) use Corollary 4. The "Equivalently" part can be easily proved using (1) - (3), by contrposition. □

Setting $* = t$ in Corollary 4 or in Corollary 5 we get a characterization of domains of finite t-character.

As a further corollary we provide a simpler proof of an important result of [7]. Recall that D is an almost GCD (AGCD) domain if for each pair x, y of nonzero elements of D there is a natural number $n = n(x, y)$ such that $x^n D \cap y^n D$ is principal. From the remark after Lemma 3.3 of [3, page 290] it follows that D is an AGCD domain if and only if for every set $x_1, x_2, ..., x_r \in D\backslash\{0\}$ there is a natural number n such that $(x_1^n, x_2^n, ..., x_r^n)_v = dD$ and from this, using the fact that $x_1, x_2, ..., x_r$ do not share a maximal t-ideal if and only if $x_1^{n_1}, x_2^{n_2}, ..., x_r^{n_r}$ do not share a maximal t-ideal for all n_i natural, one can also conclude that d is a unit if and only if $(x_1, x_2, ..., x_r)_v = D$, or in other words d is a nonunit if and only if $(x_1, x_2, ..., x_r)_v \neq D$. Let D be an AGCD domain and x a nonzero nonunit element of D. The *span* of x is the set of all nonzero nonunit elements of D dividing some power of x. In [7, Corollary 2.1(a)] it was shown that an AGCD domain is of finite t-character if and only if the span of every nonzero nonunit of D does not contain an infinite sequence of mutually v-coprime elements. Recall that x, y are v-coprime if $(x, y)_v = D$. Since, by the definition of the t-operation, for every finitely generated nonzero ideal F of a domain $F_v = F_t$, a pair of v-coprime elements is t-comaximal.

Corollary 6. *Let D be an AGCD domain and x a nonzero nonunit element of D. Then x belongs to an infinite number of maximal t-ideals if and only if the span of x contains an infinite family of mutually t-comaximal elements of D.*

Proof. The implication ($\Leftarrow$) can be shown as in the proof of Theorem 1. For the converse, let x be a nonzero nonunit that is contained in infinitely many maximal t-ideals of D. Then as in Corollary 4, for $* = t$,there is an infinite family $\{F_\alpha\}_{\alpha \in I}$ of proper t-ideals of finite type containing x. Now note that for any $\alpha \in I$ we have $F_\alpha = (z_{1\alpha}, .., z_{k_\alpha \alpha})_t = (z_{1\alpha}, .., z_{k_\alpha \alpha})_v \neq D$. But then there is a nonunit d_α and a natural number n_α such that $((z_{1\alpha})^{n_\alpha}, .., (z_{k_\alpha \alpha})^{n_\alpha})_t = d_\alpha D$. Since for each pair $\alpha, \beta \in T$ with $F_\alpha \neq F_\beta$ we have $((z_{1\alpha}, .., z_{k_\alpha \alpha}), (z_{1\beta}, .., z_{k_\beta \beta}))_t = D$ we have by [3, Lemma 3.2] $((z_{1\alpha})^{n_\alpha n_\beta}, .., (z_{k_\alpha \alpha})^{n_\alpha n_\beta}, (z_{1\beta})^{n_\alpha n_\beta}, .., (z_{k_\beta \beta})^{n_\alpha n_\beta})_t = D$ which results in $(d_\alpha^{n_\beta}, d_\beta^{n_\alpha})_t = D$ which forces $(d_\alpha, d_\beta)_t = D$. Now for each α, $x^{n_\alpha k_\alpha} \in ((z_{1\alpha}, .., z_{k_\alpha \alpha})^{n_\alpha k_\alpha})_t \subseteq (z_{1\alpha}^{n_\alpha}, .., z_{k_\alpha \alpha}^{n_\alpha})_t = d_\alpha D$, so d_α is in the span of x for each α. Since there are infinitely many mutually t-comaximal F_α there are infinitely many mutually t-comaximal elements in the span of x. $\square$

Setting $* = d$ in Corollary 4 we note that Γ in this case is the set of nonzero finitely generated proper ideals. Thus we have the following corollary.

Corollary 7. *Let D a domain and I a nonzero proper finitely generated ideal of D. Then I is contained in an infinite number of maximal ideals of D if and only if there exists an infinite family of mutually comaximal proper finitely generated ideals containing I.*

Indeed by setting $* = d$ in Corollary 5 we can get a useful d-analog of Corollary 5. Now we note that an integral domain D is not of finite character if and only if there is a finitely generated ideal I such that I is contained in infinitely many maximal ideals if and only if (by Corollary 7) there is nonzero finitely generated proper ideal I that is contained in infinitely many mutually comaximal proper finitely generated ideals. Thus we have the following form of the statement in Corollary 7.

Corollary 8. *An integral domain D is of finite character if and only if every nonzero finitely generated proper ideal of D is contained in at most a finite number of mutually comaximal finitely generated proper ideals. Consequently a Noetherian domain D is of finite character if and only if every proper nonzero ideal of D is contained in at most a finite number of proper mutually comaximal ideals of D.*

Requiring the set Γ to consist of t-invertible t-ideals and setting $* = t$ in Theorem 1 we get the following result.

Corollary 9. *Let D be a t-sub-Prüfer domain. Let I be a nonzero finitely generated ideal of D with $I_t \neq D$. Then I is contained in an infinite number of maximal t-ideals if and only if there exists an infinite family of mutually t-comaximal proper*

t-invertible t-ideals containing I. Equivalently, with the same assumptions on I, I is contained in at most a finite number of maximal t-ideals if and only if I is contained in at most a finite number of proper mutually t-comaximal t-invertible t-ideals.

It was shown in [16, Proposition 4] that if a t-invertible t-ideal in D is contained in an infinite number of mutually t-comaximal t-invertible t-ideals then there is a t-ideal in D that is t-locally principal but not t-invertible. Next let us note the following result.

Proposition 10. *Let A be a nonzero ideal, in a domain D, such that A is t-locally principal yet not t-invertible then every nonzero element of A belongs to an infinite number of maximal t-ideals.*

Proof. Suppose on the contrary that there is $x \in A\backslash\{0\}$ such that x belongs to only a finite set of maximal t-ideals, $S = \{M_1, M_2, ..., M_n\}$. Then for each $M \in t-\max(D)\backslash S$ we have $AD_M = D_M$. This gives $AD_{M_i} = a_iD_{M_i}$, where a_i can be assumed to be in A and $i = 1, 2, ..., r \leq n$.Form $B = (x, a_1, a_2, ..., a_r)$ and note that $B \subseteq A$ and so $B^w \subseteq A^w$. On the other hand for each i, $AD_{M_i} = a_iD_{M_i} \subseteq BD_{M_i}$ and so $A = A^w \subseteq B^w$. This forces $A^w = B^w$, which makes B^w t-locally principal. Since $w \leq t$ we have $A_t = B_t$, but then being t-locally principal, A is t-invertible [12, Corollary 2.7]. □

Remark 11. *From the proof of Proposition 10 we conclude that if a nonzero ideal A is t-locally principal then A^w is a t-ideal. To see this note that for any maximal t-ideal M, $AD_M = aD_M$ forces $A_t \subseteq AD_M$ which in turn forces $A_t \subseteq \bigcap_{P \in t-\max(D)} AD_P = A^w$. But generally $A^w \subseteq A_t$, because w is of finite character. Now since w is a star operation, $A^w = \bigcap_{P\in t-\max(D)} A^wD_P = \bigcap_{P\in t-\max(D)} A_tD_P$. Thus A being t-locally principal is the same as A_t being t-locally principal. This observation may be compared with the known fact that a nonzero locally principal ideal is a t-ideal* [1, Theorem 2.1].

Thus if there is in a domain D, a nonzero ideal that is t-locally principal yet not t-invertible then D is not of finite t-character. On the other hand if a t-sub-Prufer D is not of finite t-character, then by Corollary 9, there is a nonzero principal ideal xD of D that is contained in infinitely many mutually t-comaximal t-invertible t-ideals which then gives rise to a t-ideal that is t-locally principal yet not t-invertible, as in [16, Proposition 4]. In view of the above observations, Corollary 9 can be restated as follows.

Corollary 12. *Let D be a t-sub-Prüfer domain then D is not of finite t-character if and only if there is a t-ideal I in D such that I is t-locally principal yet not t-invertible. Equivalently a t-sub-Prüfer domain is of finite t-character if and only if every t-locally principal ideal of D is a t-invertible ideal.*

We can prove Corollaries 9, 12 by replacing t by $*$ of finite character, using similar procedure. Yet since, for a star operation $*$ of finite character, every $*$-invertible $*$-ideal is a t-invertible t-ideal [15, Theorem 1.1 (e)] we shall keep our attention focused on the t-operation even at the cost of going case by case. As a PVMD is t-sub-Prüfer, Corollary 12 recovers Proposition 5 of [16]. Further as a Prüfer domain is a PVMD in which every t-invertible t-ideal is actually invertible Corollary 12 also recovers the results on Prüfer domains, stated in [16], but for reference, and because Prüfer domains are better understood we shall re-write Corollary 12 as follows.

Corollary 13. *Let D be a Prüfer domain then D is not of finite character if and only if there is a nonzero ideal I in D such that I is locally principal yet not invertible. Equivalently a Prüfer domain is of finite character if and only if every nonzero locally principal ideal of D is invertible.*

Next we consider domains in which Γ consists of all proper nonzero principal integral ideals in the set of all $*$-ideals of finite type. These domains fall under $*$-sub-Prüfer and so the corresponding statements are again corollaries to the main theorem and the equivalence being obvious we will include only a plain characterization in each case.

Recall that Cohn [5] called a domain D pre-Bézout if every pair x, y of coprime elements of D is comaximal. It was shown in [14] that an atomic pre-Bézout domain is a PID [14, Corollary 6.6]. Let us call a domain D a special pre-Bézout (spre-Bézout) domain if every finite coprime set of elements generates D. Thus D is a spre-Bézout domain if and only if for each finite set $x_1, x_2, ..., x_n \in D\backslash\{0\}$ if $(x_1, x_2, ..., x_n) \subseteq dD$ implies that d is a unit then $(x_1, x_2, ..., x_n) = D$. Thus in a spre-Bézout domain every nonzero proper finitely generated ideal is contained in an integral principal ideal of D. Indeed in a spre-Bézout domain the set of proper nonzero integral principal ideals forms Γ in the set of proper nonzero finitely generated ideals.

Corollary 14. *A spre-Bézout domain D is of finite character if and only if every nonzero proper finitely generated ideal of D is divisible by at most a finite number of mutually coprime elements, if and only if every nonzero nonunit of D is divisible by at most a finite number of mutually coprime elements of D.*

The pre-Bézout property was generalized in [14] as follows: A domain D has the property λ if every two coprime elements x, y of D are v-coprime. That is $GCD(x, y) = 1$ implies that $(x, y)_v = D$. (In [14, Proposition 6.4] it was shown that an atomic λ-domain is a UFD.) This property λ can be generalized as Λ: If for $x_1, x_2, ..., x_n \in D\backslash\{0\}$ $(x_1, x_2, ..., x_n) \subseteq dD$ implies that d is a unit then $(x_1, x_2, ..., x_n)_v = D$. But this Λ-property is well known as the PSP property, where PSP stands for "primitive polynomials are superprimitive". Now note that a t-Schreier domain in which every t-invertible t-ideal is principal is what is known in the literature as a pre-Schreier domain and it is also well known that a pre-Schreier domain is a PSP domain. So, for PSP domains the set of principal integral ideals is the set Γ in proper finitely generated t-ideals. Note that a GCD domain is a pre-Schreier domain which in turn is a PSP domain and there are examples that show that a PSP domain is not necessarily pre-Schreier and a pre-Schreier is not necessarily a GCD domain. For a discussion of these notions the reader may consult [4, Section 3].

Corollary 15. *An integral domain D with PSP property is of finite t-character if and only if every nonzero nonunit of D is divisible by at most a finite number of mutually coprime nonunits.*

The term *homogeneous* has often been used in the study of generalizations of unique factorization in integral domains in which some elements may not be expressible as finite products of irreducible elements, see e.g. [2]. The sense is the same as used here, though a homogeneous element was defined in such a way that the ideal it generates is homogenous in the sense of this paper. Homogeneous elements have also been used in a recent work on factorization in Riesz groups [13], which led to the study of t-Schreier domains and the t-Schreier domains of finite t-character in [8], where a homogeneous t-invertible t-ideal made its appearance. While working on t-Schreier domains a somewhat general approach presented itself. It can be termed as the *poset approach*. The above results follow the pattern suggested by that approach. Since this approach is general we include it below for mathematicians in other areas to see if it can be of use.

Let Ω be a partially ordered set and $\emptyset \neq \Gamma \subseteq \Omega$. Say that the elements $B_1, B_2 \in \Omega$ are *comaximal* if there is no $B \in \Omega$ with $B_1, B_2 \leq B$; in this case we write $(B_1, B_2) = 1$. Assume that the following axioms hold.

(1) For each $I \in \Omega$, there exists $M \in Max(\Omega)$ (= the set of maximal elements of Ω) such that $I \leq M$.

(2) If $A_1, A_2 \in \Gamma$ and $B \in \Omega$ satisfy $A_1, A_2 \leq B$, there exists $A \in \Gamma$ such that $A_1, A_2 \leq A \leq B$.

(3) If $B_1, B_2 \in \Omega$ are comaximal, there exist $A_1, A_2 \in \Gamma$ comaximal such that $A_i \leq B_i$, $i = 1, 2$.

By axiom (1), $B_1, B_2 \in \Omega$ are comaximal if and only if there is no $M \in Max(\Omega)$ with $B_1, B_2 \leq M$. Call an element $A \in \Gamma$ *homogeneous* if $A \leq M$ for a unique maximal $M \in Max(\Omega)$.

Lemma 16. *An element $A \in \Gamma$ is homogeneous if and only if there are no $B, C \in \Gamma$ comaximal such that $A \leq B, C$.*

Proof. ($\Rightarrow$). If M is the only maximal element $\geq A$ and $B, C \in \Gamma$ such that $A \leq B, C$, then $B, C \leq M$. ($\Leftarrow$). Suppose that $A \leq M_1, M_2$ where M_1, M_2 are two distinct maximal elements. Clearly, M_1, M_2 are comaximal. By axiom (3), there exist $B_1, B_2 \in \Gamma$ comaximal such that $B_i \leq M_i$, $i = 1, 2$. By axiom (2), we can assume that $A \leq B_1, B_2$. •

□

Theorem 17. *An element $I \in \Gamma$ is $\leq$ an infinite number of maximal elements if and only if there exists an infinite family of mutually comaximal elements in Γ which are $\geq I$.*

Proof. The implication ($\Leftarrow$) follows from axiom (1) and the fact that a maximal element cannot be $\geq$ that two comaximal elements. ($\Rightarrow$). We prove the contrapositive statement. Assume that: ($\sharp$) there is no infinite family of mutually comaximal elements in Γ which are $\geq I$. First we show the following property: ($\sharp\sharp$) every $I' \in \Gamma$, $I' \geq I$, is $\leq$ some homogeneous element. Deny. As I' is not homogeneous, there exist $P_1, N_1 \in \Gamma$ comaximal such that $P_1, N_1 \geq I'$ (cf. Lemma 16). Since N_1 is not homogeneous, there exist $P_2, N_2 \in \Gamma$ comaximal such that $P_2, N_2 \geq N_1$. Note that $(P_1, P_2) = (P_1, N_2) = 1$. By induction, we can construct an infinite sequence $(P_k)_{k \geq 1}$ of mutually comaximal elements of Γ with $I' \leq P_k$, $k \geq 1$. This fact contradicts condition ($\sharp$). So ($\sharp\sharp$) holds. Now using ($\sharp$) and ($\sharp\sharp$) we can find a finite set $H_1, ..., H_n \in \Gamma$ of mutually comaximal homogeneous elements $\geq I$ such that there is no $J \in \Gamma$, $J \geq I$, such that $(J, H_i) = 1$, $1 \leq i \leq n$. Let M_i be the unique maximal element $\geq H_i$. We claim that the maximal elements $\geq I$ are

$M_1, ..., M_n$. Deny, that is, assume there exists $M \in Max(\Omega) \setminus \{M_1, ..., M_n\}$ with $M \geq I$. By axiom (3), there exist $A_i, B_i \in \Gamma$ comaximal such that $A_i \leq M$, $B_i \leq M_i$, $i = 1, ..., n$. Hence $I, A_1, ..., A_n \leq M$, so iterating axiom (2) there exists $C \in \Gamma$ with $I, A_1, ..., A_n \leq C \leq M$. By our choice of $H_1, ..., H_n$, we get that C is not comaximal to some element H_j. Since M_j is the unique maximal element $\geq H_j$, we get $C \leq M_j$, so $C, B_j \leq M_j$. Hence $A_j, B_j \leq M_j$ which is a contradiction because $(A_j, B_j) = 1$. □

Let D be a domain and $*$ be a finite character star operation on D. Theorem 1 is a particular case of Theorem 17; specifically, if we take $\Omega =$ the set of proper $*$-ideals of D and $\Gamma =$ the set of proper $*$-finite $*$-ideals of D.

We must point out that the property ($\sharp$) in Theorem 1 and in Theorem 17 is the ideal theoretic version of Conrad's F condition as described in [13] for Riesz groups. Stated for lattice ordered groups, in [6], as F: each strictly positive element is greater than or equal to at most a finite number of (mutually) disjoint elements.

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