T-SPLITTING MULTIPlicative SETS OF IDEALS IN INTEGRAL DOMAINS

GYU WHAN CHANG, TIBERIU DUMITRESCU, AND MUHAMMAD ZAFRULLAH

Abstract. Let $D$ be an integral domain. We study those multiplicative sets of ideals $S$ of $D$ with the property that every nonzero principal ideal $dD$ of $D$ can be written as $dD = (AB)_t$ with $A, B$ ideals of $D$ such that $A$ contains some ideal in $S$ and $(C + B)_t = D$ for each $C \in S$.

Let $D$ be an integral domain with quotient field $K$ and let $F(D)$ be the set of nonzero fractional ideals of $D$. Clearly, for $A \in F(D)$, $A^{-1} = D :_KA$ is again in $F(D)$. Recall that a mapping $A \mapsto A^*$ of $F(D)$ into itself is called a star operation on $D$ if the following conditions hold for all $a \in K \setminus \{0\}$ and $A, B \in F(D)$: (1) $(aD)^* = aD$, $(aA)^* = aA^*$, (2) $A \subseteq A^*$, if $A \subseteq B$, then $A^* \subseteq B^*$, and (3) $(A^*)^* = A^*$. $A$ is a $*$-ideal if $A = A^*$. For standard material about star operations, see sections 32 and 34 of [9]. Three well-known examples of star operations are the maps $A \mapsto A$ (the d-operation), $A \mapsto A_v$ (the v-operation) and $A \mapsto A_t$ (the t-operation), where $A_v = (A^{-1})^{-1}$ and $A_t = \cup \{B_v | 0 \neq B \subseteq A$ is finitely generated$\}$. Clearly, $A_v = A_t$ if $A$ is finitely generated. An ideal $A \in F(D)$ is t-invertible if $(AA^{-1})_t = D$. In this case $A$ has finite type, that is, $A_t = \{x_1, \ldots, x_n\}_t$ for some $x_1, \ldots, x_n \in A$. $D$ is called a Prüfer v-multiplication domain (PVMD), if every finitely generated ideal $A \in F(D)$ is t-invertible. The t-class group $Cl_t(D)$ of $D$ is the group of t-invertible fractional t-ideals, under the product $A * B = (AB)_t$, modulo its subgroup of principal fractional ideals.

The following concept was introduced and studied in [3]. A multiplicative subset $S$ of $D$ is said to be t-splitting, if for each $d \in D \setminus \{0\}$, $dD = (AB)_t$ for some ideals $A, B$ of $D$ with $A_t \cap S \neq \emptyset$ and $(B, s)_t = D$ for each $s \in S$. The main result of [3] asserts that $D + XD_S[X]$ is a PVMD if and only if $D$ is a PVMD and $S$ is a t-splitting set of $D$, where $D + XD_S[X]$ is the subring of $DS[X]$ consisting of those $f \in DS[X]$ with constant term in $D$. The t-splitting sets are investigated further in [6].

The main purpose of this note is to extend certain results from [3] and [6] to the case of multiplicative sets of ideals. We aim to show that by using the notion of t-splitting sets of ideals, we can explain a number of multiplicative phenomena that cannot be explained otherwise or are hard to explain. The main concept we use is that of a t-splitting set of ideals $S$ of a domain $D$ (see Definition 1). We show that many results from [3] and [6] can be stated for t-splitting sets of ideals. A characterization of $S$ being t-splitting using the $S$-transform of $D$ (see definition below) is given in Proposition 5. In Theorem 12, we show that the presence of a t-splitting set of ideals induces a natural cardinal product decomposition of the ordered monoid of fractional t-ideals of $D$ (with the t-product and ordered by reverse
inclusion). Restricting to t-prime ideals, this decomposition gives a well-behaved partition of the set of t-prime (resp. t-maximal) ideals of \( D \) (see Remark 14 and Corollary 15). Some applications for PVMDs and Krull domains are given in Propositions 16 and 17. The final part of this note contains several Nagata-type theorems.

Throughout this note, all rings are integral domains. All undefined terminology is standard as in [9]. Let \( D \) be an integral domain with quotient field \( K \), \( S \) a multiplicative set of ideals of \( D \) and \( D_S = \{ x \in K \mid xA \subseteq D \text{ for some } A \in S \} \) the \( S \)-transform of \( D \) (see [4] for basic properties of this construction). If \( I \) is an ideal of \( D \), then \( I_S = \{ x \in K \mid xA \subseteq I \text{ for some } A \in S \} \) is an ideal of \( D_S \) containing \( I \). Denote by \( S^\perp \) the set of all ideals \( B \) of \( D \) with \( (A + B)_t = D \) for all \( A \in S \). Note that \( S^\perp \) is also a multiplicative set of ideals. Call it the \( t \)-complement of \( S \). Consider also, the multiplicative set of ideals \( sp(S) \supseteq S \) consisting of all ideals \( C \) of \( D \) with \( C_t \supseteq A \) for some \( A \in S \). It is easy to see that \( sp(sp(S)) = sp(S) \), \( sp(S^\perp) = S^\perp \) and \( D_S = D_{sp(S)} \).

We begin by providing a formal definition of the notion of \( t \)-splitting sets of ideals.

**Definition 1.** Let \( S \) be a multiplicative set of ideals of \( D \) and \( S^\perp \) its \( t \)-complement. We call \( S \) a \( t \)-splitting set of ideals if every nonzero principal ideal \( dD \) of \( D \) can be written as \( dD = (AB)_t \) with \( A \in sp(S) \) and \( B \in S^\perp \).

Clearly, \( S \) is \( t \)-splitting if and only if \( sp(S) \) is \( t \)-splitting. If \( S \subseteq D \) is a saturated multiplicative set of \( D \) and \( S = \{ sD \mid s \in S \} \), then \( S \) is \( t \)-splitting in the sense of [3] if and only if \( S \) is \( t \)-splitting in our sense.

In a Krull domain \( E \), every nonzero proper principal ideal can be (uniquely) written as a \( t \)-product of height-one primes [7, Theorem 3.12], so every set of height-one prime ideals of \( E \) generates a \( t \)-splitting set (see also Proposition 17). Some easy consequences of Definition 1 are given below.

**Proposition 2.** If \( S \) is a \( t \)-splitting set of ideals of \( D \), then the following assertions hold.

(a) \( S^\perp \) is \( t \)-splitting.

(b) For every \( C \in S \), \( C_t \) contains some \( t \)-invertible ideal of \( sp(S) \).

(c) The set \( S_t \) of all \( t \)-invertible ideals in \( sp(S) \) is a \( t \)-splitting set with \( t \)-complement \( S^\perp \) and \( sp(S_t) = sp(S) \).

**Proof.** (a) is clear from Definition 1. For (b) and (c), note that when \( 0 \neq d \in C \in S \) and \( dD = (AB)_t \) with \( A \in sp(S) \) and \( B \in S^\perp \), it follows that \( A \) is \( t \)-invertible and \( C_t \supseteq A \). Indeed, as \( C \in S \) and \( B \in S^\perp \), we get \((C + B)_t = D \), so \( A \subseteq A_t = (A(C + B))_t \subseteq C_t \). So, (b) follows, and, consequently, \( sp(S_t) \supseteq sp(S) \). Thus (c) follows from the remarks accompanying Definition 1. \( \Box \)

In [8], a multiplicative set of ideals \( S \) of \( D \) is said to be \( v \)-finite if for each \( A \in S \), \( A_t \) contains some \( v \)-finite ideal \( J \in sp(S) \). Since an invertible \( t \)-ideal is \( v \)-finite, part (b) of the preceding result shows that a \( t \)-splitting set is \( v \)-finite. Our next result shows that, when \( S \) is \( t \)-splitting, the \( t \)-product decomposition imposed in Definition 1 for the principal ideals extends to all \( t \)-ideals (thus extending [3, Lemma 4.6]).

**Proposition 3.** Let \( S \) be a \( t \)-splitting set of ideals of \( D \). Then for every nonzero ideal \( I \) of \( D \), \( I_t \) can be written as \( I_t = (AB)_t \) with \( A \in sp(S) \) and \( B \in S^\perp \).
This decomposition is unique in the following sense. If \((AB)_t = (A'B')_t\) with \(A, A' \in \text{sp}(S)\) and \(B, B' \in S^\perp\), then \(A_t = A'_t\) and \(B_t = B'_t\). In particular, if \(I_t\) is of finite type, then we can choose \(A\) and \(B\) to be finite type \(t\)-ideals.

**Proof.** Let \(I\) be a nonzero ideal of \(D\) and set \(J = I \setminus \{0\}\). As \(S\) is a \(t\)-splitting set, for each \(j \in J\), we can write \(jD = (A_jB_j)_t\) with \(A_j \in \text{sp}(S)\) and \(B_j \in S^\perp\). Then \(I_t = (\bigcup jD)_t = (\bigcup (A_jB_j))_t = (\bigcup A_jB_j)_t\). But \((\bigcup A_jB_j)_t = ((j \cap h)A_h)_t\).

Indeed, the inclusion \(\subseteq\) is clear. For \(\supseteq\), let \(h, i \in J\) with \(h \neq i\). Then \((A_i + B_h)_t = D, sp A_hB_i \subseteq (A_hB_i(A_i + B_h))_t \subseteq (\bigcup A_jB_j)_t\). Finally, note that \(j A_j \in \text{sp}(S)\) and \(j B_j \in S^\perp\).

For the uniqueness part, assume that \((AB)_t = (A'B')_t\) with \(A, A' \in \text{sp}(S)\) and \(B, B' \in S^\perp\). Since \((A + B')_t = (A' + B)_t = D, we get \(A_t = (A + B')_t = (A' + B)_t = (AA' + (AB)_t)_t = (A'A' + (A'B')_t)_t = (A + B')A'_t = A'_t\). Similarly, \(B_t = B'_t\).

The “in particular” part was proved on the way. \(\Box\)

As a consequence, \(S^{\perp \perp} = \text{sp}(S)\). Indeed, let \(C\) be in the \(t\)-complement of \(S^\perp\). As shown above, \(C_t = (AB)_t\) for some \(A \in \text{sp}(S)\) and \(B \in S^\perp\). Since \((C + B)_t = D\) and \(C \subseteq B_t\), we get \(B_t = D\). So \(C_t = A_t \in \text{sp}(S)\), hence \(C \in \text{sp}(S)\).

In Proposition 5, we generalize [3, Lemma 4.2]. We need the next lemma which relies on [14, Lemma 3.4] and [8, Proposition 1.2].

**Lemma 4.** Let \(S\) be a multiplicative set of ideals of \(D\) and \(I\) a nonzero ideal of \(D\). Then

(a) \((ID)_t = (I_tD)_t\).

(b) If \(I\) is a \(t\)-invertible ideal of \(D\) and \((ID)_t = D_S\), then \(I \in \text{sp}(S)\).

**Proof.** (a) is a part of [14, Lemma 3.4]. For (b), assume that \(I\) is \(t\)-invertible. By [8, Proposition 1.2], \((JD)_t = (J_t)_S\) for each finitely generated nonzero ideal \(J\) of \(D\) with \(D : J\) \(v\)-finite. As \(I\) is \(t\)-invertible, \(I_t = J_t\) for some \(J\) \(v\)-finite and \(I \subseteq J\). Moreover, \(D : I = D : J\) \(v\)-finite and, by (a), \((ID)_t = (J_t)_S\). So, \(D_S = (ID)_t = (JD)_t = (J_t)_S = (I_t)_S\). Hence \(1 \in (I_t)_S\), that is, \(H \subseteq I_t\) for some \(H \in S\). Consequently, \(I \in \text{sp}(S)\). \(\Box\)

**Proposition 5.** Let \(S\) be a multiplicative set of ideals of \(D\). Then \(S\) is \(t\)-splitting if and only if \(S\) is \(v\)-finite and \(dD_S \cap D\) is a \(t\)-invertible ideal for each \(0 \neq d \in D\).

**Proof.** Assume that \(S\) is \(t\)-splitting. Then \(S\) is \(v\)-finite, as shown in the paragraph after Proposition 2. Let \(0 \neq d \in D\). Then \(dD = (AB)_t\) for some \(A \in S\) and \(B \in S^\perp\).

As \(B\) is \(t\)-invertible, it suffices to show that \(dD_S \cap D = B_t\). In particular, it will follow that \(dD_S \cap D \subseteq S^\perp\). As \(A(d^{-1}B_1) \subseteq d^{-1}(AB)_t = D, we get \(d^{-1}B_t \subseteq D_S\), hence \(B_t \subseteq dD_S \cap D\). On the other hand, let \(x \in dD_S \cap D\). Then \(C(d^{-1}x) \subseteq D\) for some \(C \in S\). So \(Cx \subseteq dD \subseteq B_t\), hence \(x \in B_t\), because \((C + B)_t = D\).

Conversely, assume that \(S\) is \(v\)-finite and \(dD_S \cap D\) is a \(t\)-invertible ideal for each \(0 \neq d \in D\). Let \(0 \neq d \in D\). As \(B = dD_S \cap D\) is a \(t\)-invertible ideal containing \(dD, dD = (AB)_t\) for some \(t\)-invertible ideal \(A\) of \(D\). Note that \(BD_S \subseteq dD_S\). By part (a) of Lemma 4, we get \(dD_S = (AB)_tD_S = (ABD_S)_t \subseteq (dAD_S)_t\), hence \((AD_S)_t = D_S\). By part (b) of Lemma 4, \(A \in \text{sp}(S)\). To verify that \(B \in S^\perp = \text{sp}(S)^\perp\), it suffices to show that \((B + H)_t = D\) for each \(t\)-ideal \(H \in \text{sp}(S)\). By the second part of our assumption, we may assume that \(H\) is \(v\)-finite. If \(x \in H^{-1}B^{-1}\), then \(x \in D_S\), so \(Bx \subseteq B_D_S \cap D = dD_S \cap D = B\). As \(B\) is \(t\)-invertible, \(x \in D\). Thus
\((H + B)^{-1} = H^{-1} \cap B^{-1} = D\), that is, \((H + B)_v = D\). So \((H + B)_t = (H + B)_v = D\), because \(H\) and \(B\) are \(v\)-finite ideals. Thus \(B \in S^\perp\).

To see that in the 'if' part of the preceding proposition, the assumption that \(S\) is \(v\)-finite is essential, we may use the following example from [8]. Let \(V\) be a nontrivial valuation domain whose maximal ideal \(M\) is idempotent and \(S = \{D, M\}\). Then \(V_S = V\), because \(V : M = V\). So \(dV_S \cap V\) is \(t\)-invertible for each \(0 \neq d \in V\). However, \(S\) is not \(v\)-finite.

**Remark 6.** Let \(S\) be a \(t\)-splitting set of ideals of \(D\), \(I\) a nonzero ideal of \(D_S\) and \(0 \neq d \in I \cap D\). As shown in the proof of Proposition 5, \(dD_S \cap D \in S^\perp\). Hence \(I \cap D \in S^\perp\), because \(I \cap D \supseteq dD_S \cap D\). Similarly, \(I \cap D \in sp(S)\) whenever \(I\) is a nonzero ideal of \(D_S\).

The next proposition is only a restatement, in our setup, of [3, Theorem 4.10]. The proof is virtually the same. We begin with a simple lemma.

**Lemma 7.** If \(S\) is a multiplicative set of ideals of \(D\), then \(D = D_S \cap D_{S^\perp}\).

**Proof.** Let \(x \in D_S \cap D_{S^\perp}\). Then \(xA \subseteq D\) and \(xB \subseteq D\) for some \(A \in S\) and \(B \in S^\perp\). So \(xD = x(A + B)_t = (xA + xB)_t \subseteq D\), hence \(x \in D\).

**Proposition 8.** Let \(S\) be a \(t\)-splitting set of ideals of \(D\) and \(I\) a nonzero ideal of \(D\). Then

\[I_t = (ID_S)_t \cap (ID_{S^\perp})_t = (((ID_S)_t \cap D)((ID_{S^\perp})_t \cap D))_t.\]

**Proof.** By Lemma 7, \(D = D_S \cap D_{S^\perp}\). Hence by [1, Theorem 2], the map sending a nonzero fractional ideal \(A\) of \(D\) into \(A^* = (AD_S)_t \cap (AD_{S^\perp})_t\) is a finite character star-operation on \(D\). Consequently, \(I_t \supseteq I^*\). Part (a) of Lemma 4 supplies the opposite inclusion. For the second equality, set \(U = (ID_S)_t \cap D\) and \(V = (ID_{S^\perp})_t \cap D\). By Remark 6, \(U \in S^\perp\) and \(V \in sp(S)\), so \((U + V)_t = D\). Consequently, \(I_t = U \cap V = (U \cap V)_t = (UV)_t\).

**Remark 9.** Let \(S\) be a \(t\)-splitting set of ideals of \(D\) and \(I\) a nonzero ideal of \(D\). By Proposition 3, \(I_t = (AB)_t\) with \(A \in sp(S)\) and \(B \in S^\perp\). Combining the previous result, Remark 6 and Proposition 3, we get \(A_t = (ID_S)_t \cap D\) and \(B_t = (ID_{S^\perp})_t \cap D\). Note that \((ID_S)_t \cap D\) and \((ID_{S^\perp})_t \cap D\) are \(t\)-ideals of \(D\), cf. Lemma 4 and [5, Proposition 1.1].

Let \(D\) be a domain. By definition, a \(t\)-prime ideal of \(D\) is a nonzero prime ideal of \(D\) which is also a \(t\)-ideal. It is well-known that a prime ideal which is minimal over a nonzero principal ideal is \(t\)-prime. Also, a maximal \(t\)-ideal, that is, a maximal element of the set of all proper \(t\)-ideals, is a \(t\)-prime ideal (see e.g. [12]).

**Proposition 10.** Let \(S\) be a \(t\)-splitting set of ideals of \(D\) with \(t\)-complement \(S^\perp\) and let \(P\) be a prime \(t\)-ideal of \(D\). Then \(P\) is either in \(sp(S)\) or in \(S^\perp\). Moreover, if \(P \in S^\perp\) and \(Q \subseteq P\) is a nonzero prime ideal, then \(Q \in S^\perp\). A similar assertion holds for \(sp(S)\).

**Proof.** If \(0 \neq d \in P\) and \(dD = (AB)_t\) with \(A \in S\) and \(B \in S^\perp\), then \(P \supseteq A\) or \(P \supseteq B\). So \(P \in sp(S)\) or \(P \in S^\perp\), but not both because \(P_t \neq D\). For the second part, we may assume that \(Q\) is a prime \(t\)-ideal, so \(Q \in S^\perp\), by the first part.
Lemma 11. Let $S$ be a t-splitting set of ideals of $D$. Then
(a) $(AD_S)_t = D_S$ for each $A \in sp(S)$, and
(b) $I = ((I \cap D)D_S)_t = (I \cap D)_S$ for each t-ideal $I$ of $D_S$.

Proof. $S$ is $v$-finite cf. Proposition 5, so we may apply [8, Proposition 1.8] and part (iv) of [8, Proposition 1.5] to finish the proof.

Denote by $T(D)$ the ordered monoid of fractional t-ideals of $D$ with the t-product and ordered by reverse inclusion and denote by $T_+(D)$ its positive cone, that is, $T_+(D) = \{A \in T(D) | A \subseteq D\}$. When $S$ is a multiplicative set of ideals of $D$, $T(D_S) \times_c T(D_{S^\perp})$ stands for the cardinal product of the monoids $T(D_S)$ and $T(D_{S^\perp})$. Our next result is an extension of [3, Theorem 4.12].

Theorem 12. If $S$ is a t-splitting set of ideals of $D$, the map $\alpha : T(D) \rightarrow T(D_S) \times_c T(D_{S^\perp})$, $\alpha(I) = ((ID_S)_t, (ID_{S^\perp})_t)$ is a monoid order-isomorphism.

Proof. Clearly, $\alpha$ is an order-preserving monoid homomorphism. It suffices to show that $\gamma = \alpha | T_+(D) : T_+(D) \rightarrow T_+(D_S) \times T_+(D_{S^\perp})$ is a monoid order-isomorphism. Consider the map $\beta : T_+(D_S) \times_c T_+(D_{S^\perp}) \rightarrow T_+(D)$, $\beta(I, J) = ((I \cap D)(J \cap D))_t$ (note that $I \cap D \in S^\perp$ and $J \cap D \in sp(S)$, cf. Remark 6). We prove that $\gamma$ and $\beta$ are inverse to each other. Indeed, if $A \in T_+(D)$, then $\gamma(\beta(A)) = (\alpha(A))_t = (\alpha(A))_t = A$ cf. Proposition 8. Conversely, let $(I, J) \in T_+(D_S) \times_c T_+(D_{S^\perp})$ and set $A = \beta(I, J) = ((I \cap D)(J \cap D))_t$. Since $I \cap D \in sp(S)$, $((J \cap D)D_S)_t = D_S$, cf. Lemma 11. Again by Lemma 11, $(I \cap D)_S)_t = I$. So $(AD_S)_t = ((I \cap D)_S)_t = I$. Similarly, $(AD_{S^\perp})_t = J$. Thus $\gamma(\beta(I, J)) = (I, J)$.

The next result extends [3, Remark 4.13]. Denote by $TI(D)$ the group of fractional t-invertible t-ideals of $D$ with the t-product and by $Cl_t(D)$ the t-class group of $D$, that is, the factor group of $TI(D)$ modulo its subgroup of principal fractional ideals. For $I \in TI(D)$, let $[I]$ denote the image of $I$ in $Cl_t(D)$.

Remark 13. Let $S$ be a t-splitting set of ideals of $D$. By Theorem 12, the map $\alpha$ given there induces an isomorphism $TI(D) \rightarrow TI(D_S) \times TI(D_{S^\perp})$. Moreover, if $A$ is a principal fractional ideal of $D$, then both components of $\alpha(A)$ are principal. Consequently, $\alpha$ induces a surjective group homomorphism $\tilde{\alpha} : Cl_t(D) \rightarrow Cl_t(D_S) \times Cl_t(D_{S^\perp})$, $\tilde{\alpha}(I) = ((ID_S)_t, (ID_{S^\perp})_t)$. As documented in [3, Remark 4.13], $\tilde{\alpha}$ need not be a monomorphism.

For a domain $D$, let $t$-$\text{Spec}(D)$ (resp., $t$-$\text{Max}(D)$) denote the set of all t-prime ideals (resp., maximal t-ideals) of $D$.

Remark 14. Let $S$ be a t-splitting set of ideals of $D$. From the proof of Theorem 12, we get a one-to-one correspondence between $S^\perp \cap T_+(D)$ and $T_+(D_S)$ given by $\alpha : (AD_S)_t \rightarrow (AD_S)_t$, and $I \mapsto I \cap D$. Restricting, we get a one-to-one correspondence between $S^\perp \cap t$-$\text{Spec}(D)$ and $t$-$\text{Spec}(D_S)$. By [4, Theorem 1.1], if $Q \in t$-$\text{Spec}(D_S)$, then $(D_S)_Q = D_{Q \cap D}$. Also, we get a one-to-one correspondence between $sp(S) \cap t$-$\text{Spec}(D)$ and $t$-$\text{Spec}(D_{S^\perp})$. Note that by Proposition 10, the sets $sp(S) \cap t$-$\text{Spec}(D)$ and $S^\perp \cap t$-$\text{Spec}(D)$ give a partition of $t$-$\text{Spec}(D)$. Similar correspondences hold when replacing $t$-$\text{Spec}$ by $t$-$\text{Max}$.

Therefore, by Remark 14 and [4, Theorem 1.1], $t$-$\text{Max}(D_{S^\perp}) = \{P_{S^\perp} : P \in sp(S) \cap t$-$\text{Max}(D)\}$ and $(D_{S^\perp})_{P_{S^\perp}} = D_P$ for each $P \in sp(S) \cap t$-$\text{Max}(D)$. Similarly,
t-Max(D_S) = \{P_S; P \in S^\perp \cap t-Max(D)\} and (D_S)_{P_S} = D_P for each P \in S^\perp \cap t-Max(D).

Corollary 15. Let S be a t-splitting set of ideals of D. Then DS = \bigcap\{D_P| P \in t-Max(D)\} \cap S^\perp and D_{S^\perp} = \bigcap\{D_P| P \in t-Max(D) \cap sp(S)\}.

Proof. By the preceding paragraph, D_{S^\perp} = \bigcap\{(D_{S^\perp})_Q| Q \in t-Max(D_{S^\perp})\} = \bigcap\{D_P| P \in t-Max(D) \cap sp(S)\}. The other equality can be proved similarly. \Box

Let us recall from [10] that D is a PVMD if and only if D_P is a valuation domain for each maximal t-ideal of D.

Proposition 16. Let S be a t-splitting set of ideals of D. Then every finite type t-ideal in sp(S) is t-invertible if and only if D_{S^\perp} is a PVMD.

Proof. \((\Rightarrow)\) Let Q \in t-Max(D_{S^\perp}) and P = Q \cap D. Then P \in t-Max(D) \cap sp(S) by Lemmas 4 and 11.

Let J be a nonzero finitely generated ideal of D_P. Then J = ID_P where I is a finitely generated ideal of D. Then I_t = (AB)_t for some A \in sp(S) and B \in S^\perp. Since P \in sp(S), B \not\subseteq P, and so (ID_P)_t = (I_tD_P)_t = ((AB)_tD_P)_t = (AD_P)_t. Also, since I is finitely generated, I_t, and hence A_t is of finite type; so A_t is t-invertible. Note that P is a prime t-ideal of D; so AA^{-1} \not\subseteq P. Hence AD_P and ID_P are invertible, and thus ID_P is principal. So D_P is a valuation domain. Thus as D_P \subseteq (D_{S^\perp})_Q, (D_{S^\perp})_Q is a valuation domain, and thus D_{S^\perp} is a PVMD.

\((\Leftarrow)\) Let I \subseteq sp(S) be a finite type t-ideal of D, and let P \in t-Max(D). If P \not\subseteq sp(S), then I \not\subseteq P, and hence ID_P = D_P. Assume that P \subseteq sp(S). Then P_{S^\perp} is a t-ideal of D_{S^\perp} and D_P = (D_{S^\perp})_{P_{S^\perp}}. Since D_{S^\perp} is a PVMD, D_P is a valuation domain. Also, since I is a finite type t-ideal, ID_P is principal. Hence I is t-locally principal, and thus I is t-invertible, cf. [14, Proposition 2.6]. \Box

Our next result is a variant of [6, Theorem 2.2].

Proposition 17. Let \Gamma be a collection of t-invertible prime t-ideals of D and S the multiplicative set generated by \Gamma. Then the following statements are equivalent.

(a) S is a t-splitting set.
(b) \bigcap_{i=1}^{n}P_1 \cdots P_n = 0 for each sequence \( (P_n) \) of elements of \Gamma.
(c) D_{S^\perp} is a Krull domain.

Proof. Clearly, S^\perp is the set of ideals I of D contained in no P \in \Gamma. Note that \Gamma \subseteq t-Max(D) cf. [13, Proposition 1.3].

(a) \Rightarrow (c) Let Q \in t-Max(D) \cap sp(S) and Q' \subseteq Q a minimal prime of a principal ideal. Then Q' is a t-ideal and Q' \subseteq sp(S) cf. Proposition 10. Then Q' \supseteq P_1 \cdots P_n for some P_i \in \Gamma. Hence Q' = P_i = Q because P_i \in t-Max(D). Thus t-Max(D) \cap sp(S) = \Gamma and each P \in \Gamma has height one. By Lemma 4, P_{S^\perp} is t-invertible in D_{S^\perp} for each P \in \Gamma. By the paragraph after Remark 14, t-Max(D_{S^\perp}) = \{P_{S^\perp}| P \in \Gamma\} and each P_{S^\perp} has height one, because (D_{S^\perp})_{P_{S^\perp}} = D_P. By [15, Theorem 3.6], D_{S^\perp} is a Krull domain.

(c) \Rightarrow (b) Let \( (P_n) \) be a sequence of elements of \Gamma and P = P_n for some n. Clearly P \not\in S^\perp. As P is t-invertible, we have (PD_{S^\perp})_t = P_{S^\perp} (see the proof of Lemma 4), so P_{S^\perp} is a prime t-ideal of D_{S^\perp}. Since D_{S^\perp} is a Krull domain, we get \bigcap_{i=1}^{n}P_1 \cdots P_n \subseteq \bigcap_{i=1}^{n}(P_i)_{S^\perp} \cdots (P_n)_{S^\perp} = 0.

(b) \Rightarrow (a) Assume that \bigcap_{i=1}^{n}P_1 \cdots P_n = 0 for each sequence \( (P_n) \) of ideals of \Gamma. Let 0 \neq d \in D. Since each P \in \Gamma is t-invertible, if I is a nonzero ideal contained in P,
we get \( I_t = (PJ)_t \), with \( J = P^{-1}I \). We use repeatedly this factorization property starting with \( I = dD \). By our assumption on \( \Gamma \), we get \( dD = (P_1 \cdots P_n J)_t \) for some \( P_1, \ldots, P_n \in \Gamma \), \( n \geq 0 \) and some ideal \( J \) contained in no \( P \in \Gamma \), thus \( J \in S^\perp \). \( \square \)

We recall that a Mori domain is a domain satisfying the ascending chain condition on integral divisorial ideals.

**Corollary 18.** A collection of \( t \)-invertible prime \( t \)-ideals of a Mori domain generates a \( t \)-splitting set.

**Corollary 19.** A collection of \( t \)-invertible uppers to zero in \( D[X] \) generates a \( t \)-splitting set.

Recall that with the realization of the power of splitting sets came various extensions of Nagata’s theorem for UFD’s (see e.g. [2]). Now the question is what can the \( t \)-splitting sets of ideals do for us? In fact they can deliver a somewhat modified version of Nagata type Theorems.

An integral domain \( D \) is said to be of **finite \( t \)-character** if every nonzero nonunit of \( D \) belongs to only finitely many maximal \( t \)-ideals of \( D \).

**Proposition 20.** Let \( S \) be a \( t \)-splitting set of ideals of an integral domain \( D \), and suppose that every proper ideal in \( S \) is contained in at most a finite number of maximal \( t \)-ideals of \( D \). Then \( D_S \) is a ring of finite \( t \)-character if and only if \( D \) is a ring of finite \( t \)-character.

**Proof.** By Proposition 10 and the paragraph preceding Corollary 15, if \( P \) is a maximal \( t \)-ideal of \( D \), then either \( P \in \text{sp}(S) \) or \( P \in S^\perp \) and that \( t \)-Max\( (D_S) = \{P_S|P \in S^\perp \cap t\text{-Max}(D)\} \). For \( 0 \neq d \in D \), let \( dD = (AB)_t \), where \( A \in \text{sp}(S) \) and \( B \in S^\perp \). Since \( A \in S \), there are only a finite number of maximal \( t \)-ideals in \( \text{sp}(S) \) containing \( A \) (and hence \( d \)). Moreover, since \( t \)-Max\( (D_S) = \{P_S|P \in S^\perp \cap t\text{-Max}(D)\} \), the number of maximal \( t \)-ideals in \( S^\perp \) containing \( d \) is finite. Therefore, \( D \) is of \( t \)-finite character. The converse is straightforward from the above observation. \( \square \)

This result can be put to direct use in a number of situations. In the following, we address a few of them.

**Corollary 21.** Let \( D \) be an integral domain and let \( S \) be a \( t \)-splitting set of ideals of \( D \) generated by height-one prime ideals. Suppose that every proper ideal in \( S \) is contained in at most a finite number of maximal \( t \)-ideals of \( D \). Then \( D_S \) is a ring of finite \( t \)-character if and only if \( D \) is a ring of finite \( t \)-character.

An integral domain \( D \) is called a **weakly Krull domain** if \( D = \cap_{P \in X(D)} D_P \) and this intersection has finite character. According to [11], a ring of \( \text{Krull type} \) is an integral domain which is a locally finite intersection of essential valuation overrings. The ring \( D \) of Krull type is an **independent ring of \( \text{Krull type} \)** if each prime \( t \)-ideal of \( D \) lies in a unique maximal \( t \)-ideal and a **generalized Krull domain** if \( D \) is weakly Krull.

**Corollary 22.** Let \( \mathcal{F} \) be a family of height-one \( t \)-invertible prime \( t \)-ideals of an integral domain \( D \). Let \( S \) be a multiplicative set of ideals generated by \( \mathcal{F} \) and suppose that every nonzero nonunit of \( D \) belongs to at most a finite number members of \( \mathcal{F} \).

1. \( D \) is a weakly Krull domain if and only if \( D_S \) is.
2. \( D \) is a generalized Krull domain if and only if \( D_S \) is.
3. \( D \) is a ring of Krull type if and only if \( D_S \) is.
\( D \) is an independent ring of Krull type if and only if \( D_S \) is.

\( D \) is a PVMD if and only if \( D_S \) is.

**Proof.** The proof consists in noting that every \( t \)-invertible prime \( t \)-ideal \( P \) is a maximal \( t \)-ideal \([13, \text{Proposition 1.3}]\) and that \( P \) being of height-one implies that \( D_P \) is a discrete valuation domain. The rest depends upon recalling the definitions of the respective notions.

In this vein it would be interesting to record the following result.

**Corollary 23.** Let \( X \) be an indeterminate over the integral domain \( D \) and \( S = \{ f \in D[X]| A_f^{-1} = D \} \). Then \( D \) is a ring of Krull type if and only if \((D[X])_S \) is a Bezout domain of finite character.

**Proof.** Recall that \( D \) is a PVMD if and only if \( D[X]_S \) is a Bezout domain \([14, \text{Theorem 3.7}]\) and that \( D \) is of finite character if and only if \( D[X] \) is \([9, \text{Exercise 1, pp.537}]\). So the result follows from Corollary 22(4) because the set \( S := \{ I \subseteq D[X]| I \text{ is an ideal of } D[X] \text{ such that } f \in I \text{ for some } f \in S \} \) is a \( t \)-splitting set of ideals.

Just to give an idea of how these results can be extended we state the following. Let \( \ast \) be a star operation on an integral domain \( D \), and let \( \ast_s \) be the finite type star operation induced by \( \ast \), i.e., \( \ast_s = \cup \{ F^s| F \subseteq I \text{ is finitely generated} \} \) for any \( I \in F(D) \). Then \( D \) is called a Pr"ufer \( \ast \)-multiplication domain if every finitely generated ideal of \( D \) is \( \ast_s \)-invertible. It is clear that Pr"ufer \( \ast \)-multiplication domains are PVMDs because \( \ast_s \subseteq I \).

**Proposition 24.** Let \( D \) be a domain, \( \ast \) a star operation of finite type on \( D \), \( F \) a family of maximal height-one principal primes of \( D \) and \( S \) the multiplicative set generated by \( F \). Suppose that each nonzero nonunit of \( D \) is contained in at most a finite number of members of \( F \). Then \( D \) is of \( \ast \)-finite character (resp., a Pr"ufer \( \ast \)-multiplication domain) if and only if \( D_S \) is of \( \ast \)-finite character (resp., a Pr"ufer \( \ast \)-multiplication domain).

We note that if the finite character star operation \( \ast \) is the identity star operation \( d \) that takes \( A \mapsto A \) for all \( A \in F(D) \), then a Pr"ufer \( \ast \)-multiplication domain is a Pr"ufer domain. Thus for \( \ast = d \) Proposition 24 gives us the following corollary.

**Corollary 25.** Let \( D \) be domain, \( F \) a family of height-one principal primes that are also maximal ideals and \( S \) the multiplicative set generated by \( F \). Suppose that every nonzero nonunit of \( D \) belongs to at most a finite number of members of \( F \). Then \( D \) is a Pr"ufer domain of finite character if and only if \( D_S \) is a Pr"ufer domain of finite character.

**Acknowledgements**

We thank the referee for several helpful suggestions.

**References**


(Chang) Department of Mathematics, University of Incheon, Incheon 402-749, Korea
E-mail address: whang@incheon.ac.kr

(Dumitrescu) Facultatea de Matematica, Universitatea Bucuresti, Str. Academiei 14, Bucharest, RO-010014, Romania
E-mail address: tiberiu@al.math.unibuc.ro

(Zafrullah) Department of Mathematics, Idaho State University, Pocatello, ID 83209-8085, USA (Current address: 57 Colgate Street, Pocatello, ID 83201)
E-mail address: mzafrullah@usa.net