Weakly Krull Inside Factorial Domains

D. D. ANDERSON, The University of Iowa, Department of Mathematics, Iowa City, Iowa 52242, dan-anderson@uiowa.edu

MUHAMMED ZAFRULLAH, Idaho State University, Department of Mathematics, Pocatello, ID 83209-8085, mzafrullah@usa.net

GYU WHAN CHANG, The University of Incheon, Department of Mathematics, Incheon, 402-748 Korea, whan@incheon.ac.kr

1 Introduction

Chapman, Halter-Koch, and Krause [8] introduced the notion of an inside factorial monoid and integral domain. Throughout we will confine ourselves to the integral domain case, but the interested reader may easily supply definitions and proofs for the monoid case. An integral domain D is *inside factorial* if there exists a divisor homomorphism φ: $F \to D^* = D - \{0\}$ where F is a factorial monoid and for each $x \in D^*$, there exists an $n \geq 1$ with $x^n \in \varphi(F)$. They showed [8, Proposition 4] that an inside factorial domain may be defined in terms of a Cale basis and it will be this characterization that we use for the definition of an inside factorial domain. A subset $Q \subseteq D^*$ is a *Cale basis for* D if $\langle Q \rangle = \{uq_{\alpha_1} \cdots q_{\alpha_n} \mid u \in U(D),\, q_{\alpha_i} \in Q\}$ is a factorial monoid with primes Q and for each $d \in D^*$ there exists an $n \geq 1$ with $d^n \in \langle Q \rangle$. Here $U(D)$ is the group of units of D. A domain D is inside factorial if and only if D has a Cale basis. They showed [8, Theorem 4] that D is inside factorial if and only if $\overline{D}$, the integral closure of D, is a generalized Krull domain with torsion t-class group $Cl_t(D)$, for each $P \in X^{(1)}(\overline{D})$, the valuation domain $\overline{D}_P$ has value group order-isomorphic to a subgroup of $(\mathbb{Q}, +)$ (we say that D is rational), and $D \subseteq \overline{D}$ is a root extension (i.e., for each $x \in \overline{D}$, there exists an $n \geq 1$ with $x^n \in \overline{D}$).

Recall that an integral domain D is *weakly Krull* [4] if $D = \bigcap_{P \in X^{(1)}(D)} D_P$ where the intersection is locally finite. While an integrally closed inside factorial domain is weakly Krull, an inside factorial domain need not be weakly Krull (see Example 3.2 below).

The purpose of this paper is threefold. First we show (Theorem 3.1) that an inside factorial domain has a unique minimal root extension which is weakly Krull (and necessarily inside factorial). Second, we give (Theorem 3.3) a number of characterizations of weakly Krull inside factorial domains. For example, we show that for an inside factorial domain the following conditions are equivalent: (1) D is weakly Krull, (2) D is an almost GCD domain, (3) t-dim $D = 1$, and (4) for each element q of a Cale basis for D, (q) is primary. Third, Theorems 3.1 and 3.3 are then used to give a new proof that if D is inside factorial, then $\overline{D}$ is a rational generalized Krull domain with torsion t-class group and $D \subseteq \overline{D}$ is a root extension.

2 Preliminaries

In this section we review some results on Cale bases, AGCD domains, and root extensions.

Let D be an inside factorial domain with Cale basis Q. Now by definition the elements of Q are primes in the monoid $\langle Q \rangle$ and the map $Q \to X^{(1)}(D)$ given by $q \to \sqrt{(q)}$ is a bijection [8, Theorem 2]. (In fact, if $Q' \subseteq D^*$ with bijection as above, then Q' is a Cale basis.) However, the elements of Q need not be irreducible nor primary as elements of D.

Also, recall that an integral domain D is an *almost GCD domain* (*AGCD domain*) [10] if for $x, y \in D^*$, there exists an $n \geq 1$ with $x^n D \cap y^n D$, or equivalently $(x^n, y^n)_v$, principal. An AGCD domain D has torsion t-class group [5, Theorem 3.4], $D \subseteq \overline{D}$ is a root extension [10, Theorem 3.1], and $\overline{D}$ is a PVMD with torsion t-class group [10, Theorem 3.4 and Corollary 3.8]. Now a PVMD with torsion t-class group is an AGCD domain [10, Theorem 3.9], but if D is an integral domain with $D \subseteq \overline{D}$ a root extension and $\overline{D}$ a PVMD with torsion t-class group, or equivalently, $\overline{D}$ is an AGCD domain, D need not be an AGCD domain. The following example is [3, Theorem 3.1] (also see Example 3.2 of this paper). Let $K \subsetneq L$ be a purely inseparable field extension and let $D = K + (\mathbf{X})L[\mathbf{X}]$ where $\mathbf{X}$ is a set of indeterminates with $|\mathbf{X}| > 1$. Then $D \subsetneq \overline{D} = L[\mathbf{X}]$ is a root extension and $L[\mathbf{X}]$ is a UFD (and hence an AGCD domain), but D is not an AGCD domain. Note that if $|\mathbf{X}| = 1$, D is an AGCD domain.

The following results on root extensions will be frequently used. (1) [5, Theorem 2.1], [8, Proposition 5] Let $R \subseteq S$ be a root extension of commutative rings. The map $\operatorname{Spec}(S) \to \operatorname{Spec}(R)$ given by $Q \to Q \cap R$ is an order isomorphism and homeomorphism with inverse given by $P \to \sqrt{P} = \{s \in S \mid s^n \in P \text{ for some } n \geq 1\}$. Moreover, $R_P \subseteq S_Q$ is a root extension. (2) [8, Proposition 5] Let $D \subseteq S$ be a root extension of integral domains. Then D is inside factorial if and only if S is inside factorial. Of course, in the case where D is inside factorial and $S \subseteq K$, the quotient field of D, $D \subseteq S$ is a root extension if and only if $D \subseteq S$ is integral.

We have remarked that an inside factorial domain need not be weakly Krull. The previously mentioned example $D = K + (\mathbf{X})L[\mathbf{X}]$ where $K \subsetneq L$ is purely inseparable and $|\mathbf{X}| > 1$ is such an example, see Example 3.2 below. In [6, Proposition 2.4] we showed that the following conditions are equivalent for an inside factorial domain D with Cale basis Q: (1) for each $q \in Q$, $\sqrt{(q)}$ is a maximal t-ideal, (2) D is weakly Krull, (3) D is an AGCD domain, (4) each $q \in Q$ is a primary element, and (5) distinct elements of Q are v-coprime. See Theorem 3.3 below where these and several more equivalences are given.

3 Weakly Krull Inside Factorial Domains

We begin this section by showing that an inside factorial domain has a unique minimal root extension which is weakly Krull. We then give an example of an inside factorial domain that is not weakly Krull and give a number of characterizations of weakly Krull inside factorial domains.

Theorem 3.1. *Let D be an inside factorial domain and let $D^{\#} = \bigcap_{P \in X^{(1)}(D)} D_P$.*
(1) $D \subseteq D^{\#}$ is a root extension and hence $D^{\#}$ is inside factorial.
(2) $D^{\#}$ is weakly Krull.
(3) Let S be an integral domain containing D such that S is weakly Krull and $D \subseteq S$ is a root extension. Then $D^{\#} \subseteq S$.
(4) Suppose that S is an integral domain with $D^{\#} \subseteq S$ a root extension. Then S is a weakly Krull inside factorial domain.

Proof. Here D is an inside factorial domain with quotient field K. Let $Q = \{q_\alpha\}$ be a Cale basis for D and let $P_\alpha = \sqrt{(q_\alpha)}$. So $Q \to X^{(1)}(D)$ given by $q_\alpha \to P_\alpha$ is a bijection.

(1) Let $0 \neq x \in D^{\#}$. Write $x = a/b$ where $a, b \in D^*$. Since suitable powers of a and b lie in $\langle Q \rangle$, we can choose $n \geq 1$ with $x^n = \epsilon q_{\alpha_1}^{n_1} \cdots q_{\alpha_s}^{n_s}$ where $\epsilon \in U(D)$ and each $n_i \in \mathbb{Z}^*$. Suppose that some $n_i < 0$. With a change of notation, if necessary, we can assume that $n_1, \cdots, n_i < 0$ and $n_{i+1}, \cdots, n_s > 0$. Now $x^n \in D_{P_{\alpha_i}}$, so $\epsilon q_{\alpha_1}^{n_1} \cdots q_{\alpha_s}^{n_s} = r/t$ where $r \in D$ and $t \in D - P_{\alpha_i}$. Then $\epsilon t q_{\alpha_{i+1}}^{n_{i+1}} \cdots q_{\alpha_s}^{n_s} = r q_{\alpha_1}^{-n_1} \cdots q_{\alpha_i}^{-n_i} \in P_{\alpha_i}$. But no factor on the left hand side lies in P_{α_i}, a contradiction. Hence each $n_i > 0$, so $x^n \in D$. Since $D \subseteq D^{\#}$ is a root extension, $D^{\#}$ is inside factorial.

(2) For $N \in X^{(1)}(D^{\#})$, let $S = D^{\#} - N$ and $P = N \cap D$. Since $D \subseteq D^{\#}$ is a root extension, $P \in X^{(1)}(D)$. Now $D^{\#} \subseteq D_P$ gives $D^{\#}_N \subseteq (D_P)_S = D_P$ where the equality follows since $\operatorname{ht} P_P = 1$ and $P_P \cap S = \varnothing$. But certainly $D_P \subseteq D^{\#}_N$, so $D_P = D^{\#}_N$. Thus $D^{\#} \subseteq \bigcap_{N \in X^{(1)}(D^{\#})} D^{\#}_N = \bigcap_{P \in X^{(1)}(D)} D_P = D^{\#}$ and hence $D^{\#} = \bigcap_{N \in X^{(1)}(D^{\#})} D^{\#}_N$. Let $0 \neq x \in K$. Then x is a unit in almost all the D_{P_α}'s. This follows since $0 \neq a \in D$ is in $P_\alpha = \sqrt{(q_\alpha)}$ if and only if $a^n \in \langle Q \rangle$ has q_α as one of its factors and thus a is in only finitely many P_α. Since each $D^{\#}_N$ is equal to a unique D_{P_α}, x is a unit in almost all the $D^{\#}_N$'s. Hence $D^{\#}$ is weakly Krull.

(3) Suppose that $D \subseteq S$ is a root extension and that $S = \bigcap_{N \in X^{(1)}(S)} S_N$. For $N \in X^{(1)}(S)$, $P = N \cap D \in X^{(1)}(D)$; so $D_P \subseteq S_N$. Then $D^{\#} = \bigcap_{P \in X^{(1)}(D)} D_P \subseteq \bigcap_{N \in X^{(1)}(S)} S_N = S$.

(4) Suppose that $D^{\#} \subseteq S$ is a root extension. Then $D \subseteq S$ is also a root extension. Hence Q is a Cale basis for $D^{\#}$ and S. Since $D^{\#}$ is weakly Krull, $\sqrt{q_\alpha D^{\#}} \in X^{(1)}(D^{\#})$ gives that $q_\alpha D^{\#}$ is $\sqrt{q_\alpha D^{\#}}$-primary. We claim that $q_\alpha S$ is $\sqrt{q_\alpha S}$-primary. Suppose that $xy \in q_\alpha S$ and $x \notin \sqrt{q_\alpha S}$. Choose $n \geq 1$ with $x^n, y^n \in D^{\#}$ and $x^n y^n \in q_\alpha D^{\#}$. Then $x^n \notin \sqrt{q_\alpha D^{\#}}$, so for some $m \geq 1$, $(y^n)^m \in q_\alpha D^{\#} \subseteq q_\alpha S$. Thus S has a Cale basis $Q = \{q_\alpha\}$ with each $q_\alpha S$ primary. By the remarks preceding Theorem 3.1, S is a weakly Krull inside factorial domain. □

Of course, for D inside factorial, $D = D^{\#}$ if and only if D is weakly Krull. Note that by [8, Proposition 5(f)] D is tamely inside factorial if and only if $D^{\#}$ is. We next give an example of an inside factorial domain that is not weakly Krull.

Example 3.2. Let $D = K + (\mathbf{X})L[\mathbf{X}]$ where $K \subseteq L$ is a purely inseparable field extension and $\mathbf{X}$ is a nonempty set of indeterminates over L. Then $D \subseteq \overline{D} = L[\mathbf{X}]$ is a root extension since $K \subseteq L$ is purely inseparable and $\overline{D}$ being factorial is inside factorial. Thus D is inside factorial. Alternatively, if we take Q to be a complete set of nonassociate prime elements of $K[\mathbf{X}]$, then Q is a Cale basis for D and hence D is inside factorial. Suppose that $|\mathbf{X}| = 1$. Then $\dim D = 1$, so $D = D^{\#}$ and hence D is weakly factorial. Suppose that $|\mathbf{X}| > 1$. We claim that $D^{\#} = L\,|\mathbf{X}|$. Let P be a height-one prime ideal of D; so $P = fL[\mathbf{X}] \cap D$ where $f \in L[\mathbf{X}]$ is irreducible. Choose $g \in D - P$ with g(0)=0. *(*If $|\mathbf{X}| = 1$ and $f = X$, we can't choose such a g. But suppose $|\mathbf{X}| > 1$. If $f \in \mathbf{X}$, take any $g \in \mathbf{X} - \{f\}$ while if $f \notin \mathbf{X}$; take $g = X \in \mathbf{X}$.*)* Now for $a \in L$, $a = ag/g \in D_P$. Hence $L \subseteq D^{\#}$, so $D^{\#} = L[\mathbf{X}]$.

We next give a number of conditions equivalent to D being a weakly Krull inside factorial domain. But first we need some more definitions. An integral domain D is *almost weakly factorial* [4] if some power of each nonunit of D is a product of primary elements. By [4, Theorem 3.4], D is almost weakly factorial if and only if D is weakly Krull and $Cl_t(D)$ is torsion. An integral domain D is a *generalized weakly factorial domain* [7] if each nonzero prime ideal of D contains a nonzero primary element. Thus an almost weakly factorial domain is a generalized weakly factorial domain. Two elements x and y of an integral domain D are *power associates* if there exist $n, m \geq 1$ and a unit $\epsilon \in U(D)$ with $x^n = \epsilon y^m$. Given a nonzero prime ideal P of D, b is a *base for* P if $P = \sqrt{(b)}$. Thus if (b) is P-primary, b is a base for P. Finally, t-$\dim D = 1$ if every prime t-ideal of D is a maximal t-ideal.

Theorem 3.3. *For an integral domain D the following conditions are equivalent.*
(1) D is inside factorial and weakly Krull.
(2) D is inside factorial and AGCD.
(3) D is inside factorial and t-$\dim D = 1$.
(4) D is inside factorial with a Cale basis Q such that $\sqrt{qD}$ is a maximal t-ideal for each $q \in Q$.
(5) D is inside factorial and for each Cale basis Q for D, $\sqrt{qD}$ is a maximal t-ideal for each $q \in Q$.
(6) D is inside factorial with a Cale basis Q such that each $q \in Q$ is primary.
(7) D is inside factorial and for each Cale basis Q for D and each $q \in Q$, q is primary.
(8) D is inside factorial with a Cale basis Q such that distinct elements of Q are v-coprime.
(9) D is inside factorial and for each Cale basis Q for D distinct elements of Q are v-coprime.
(10) D is almost weakly factorial and two nonzero primary elements of D are either v-coprime or power associates.
(11) D is a weakly Krull AGCD domain and two nonzero primary elements of D are either v-coprime or power associates.
(12) D is a generalized weakly factorial domain such that for each height-one prime ideal P of D, every pair of base elements of P are power associates and for each nonzero $x \in P$, there is an $n \geq 1$ with $x^n = db$ where b is a base for P and $d \notin P$.

Proof. The equivalence of (1), (2), and (4)–(9) is given by [6, Proposition 2.4]. (3)$\Longleftrightarrow$(4) For D inside factorial, $X^{(1)}(D) = \{\sqrt{qD} \mid q \in Q\}$ for any Cale basis Q

for D. Since a height-one prime ideal is a t-ideal, the equivalence follows. (6)$\Longrightarrow$(10) Let x be a nonzero nonunit of D. Then some power $x^n \in \langle Q \rangle$. So x^n is a product of primary elements and hence D is almost weakly factorial. Suppose that q_1 and q_2 are nonzero primary elements of D. If $\sqrt{(q_1)} \neq \sqrt{(q_2)}$, then $(q_1, q_2)_v = D$ (for say by (6)$\Longrightarrow$(3) $\sqrt{(q_1)}$ and $\sqrt{(q_2)}$ are maximal t-ideals). So suppose $\sqrt{(q_1)} = \sqrt{(q_2)} = \sqrt{(q)}$ where $q \in Q$. Then $q_1^{n_1} = \epsilon_1 q^{m_1}$ and $q_2^{n_2} = \epsilon_2 q^{m_2}$ for some $n_1, n_2, m_1, m_2 \geq 1$ and units ϵ_1, ϵ_2. But then q_1 and q_2 are power associates. (10)$\Longrightarrow$(6) Since D is almost weakly factorial, D is weakly Krull and each height-one prime P_α contains a P_α-primary element q_α. Let $Q = \{q_\alpha\}$. We claim that Q is a Cale basis for D. If $\epsilon_1 q_{\alpha_1}^{n_1} \cdots q_{\alpha_s}^{n_s} = \epsilon_2 q_{\alpha_1}^{m_1} \cdots q_{\alpha_s}^{m_s}$ where ϵ_1, ϵ_2 are units and $n_i, m_i \geq 0$, then $q_{\alpha_i}^{n_i} D_{P_{\alpha_i}} = \epsilon_1 q_{\alpha_1}^{n_1} \cdots q_{\alpha_s}^{n_s} D_{P_{\alpha_i}} = \epsilon_2 q_{\alpha_1}^{m_1} \cdots q_{\alpha_s}^{m_s} D_{P_{\alpha_i}} = q_{\alpha_i}^{m_i} D_{P_{\alpha_i}}$; so $n_i = m_i$. Hence $\langle Q \rangle$ is a factorial monoid. Let x be a nonzero nonuit of D. Then some power x^n is a product of primary elements. But each primary element in this factorization is power associate to some q_α. Thus raising x^n to an appropriate power gives that some power of x is in $\langle Q \rangle$. (11)$\Longrightarrow$(10) It suffices to show that an AGCD weakly Krull domain is almost weakly factorial. But we have already remarked that an AGCD domain has torsion t-class group and that a domain is almost weakly factorial if and only if it is a weakly Krull domain with torsion t-class group. (6)$\Longrightarrow$(11) Since (6)$\Longrightarrow$(1), D is weakly Krull and since (6)$\Longrightarrow$(2), D is an AGCD domain. And by (6)$\Longrightarrow$(10) two nonzero primary elements are either v-coprime or power associates. (10)$\Longrightarrow$(12) Certainly an almost weakly factorial domain is a generalized weakly factorial domain. Now an almost weakly factorial domain is weakly Krull and in a weakly Krull domain an element x with $\sqrt{(x)} = P'$, P' a height-one prime, is primary (for $(x) = x \bigcap_{P \in X^{(1)}(D)} D_P = \bigcap_{P \in X^{(1)}(D)} x D_P = x D_{P'} \cap D$). Thus two bases for a height-one prime ideal P are P-primary and hence by hypothesis are power associates. Finally, let $0 \neq x \in P$. Since D is almost weakly factorial, some x^n is a product of primary elements, say $x^n = q_1 \cdots q_i q_{i+1} \cdots q_m$ where $q_1, \cdots, q_i \in P$ (with necessarily $i \geq 1$) and $q_{i+1}, \cdots, q_m \notin P$. Now $(q_1 \cdots q_i)$ is still P-primary (for D is weakly Krull and $\sqrt{(q_1 \cdots q_i)} = P$). Take $b = q_1 \cdots q_i$ and $d = q_{i+1} \cdots q_n$; so $x^n = db$ where $d \notin P$ and b is a base for P. (12)$\Longrightarrow$(1) A generalized weakly factorial domain is weakly Krull [7, Corollary 2.3]. For each $P \in X^{(1)}(D)$ choose a base element x_P for P. We claim that $Q = \{x_P\}$ is a Cale basis for D; and hence D is inside factorial. As in the proof of (10)$\Longrightarrow$(6), $\langle Q \rangle$ is a factorial monoid. Let x be a nonzero nonunit of D. Let $P_1, \cdots, P_m$ be the height-one prime ideals of D containing x. By hypothesis, there is an $n \geq 1$ so that $x^n = db$ where b is a base for P_1 and $d \notin P_1$. Now b and x_{P_1} are power associates, say $b^r = u x_{P_1}^s$ where $r, s \geq 1$ and u is a unit. Then $x^{nr} = d^r u x_{P_1}^s = x_1 x_{P_1}^s$ where $x_1 \notin P_1$. Now $P_2, \cdots, P_m$ are the height-one prime ideals containing x_1. By induction, we have $x_1^t = \beta x_{P_2}^{s_2} \cdots x_{P_m}^{s_m}$ where β is a unit and $t, s_2, \cdots, s_m \geq 1$. Hence $x^{tnr} \in \langle Q \rangle$. $\square$

4 Miscellaneous Results

We have already remarked that an integral domain D is inside factorial if and only if $\overline{D}$ is a rational generalized Krull domain with torsion t-class group and $D \subseteq \overline{D}$ is a root extension [8]. We begin this section by giving an alternative proof of the implication ($\Longrightarrow$) based on Theorems 3.1 and 3.3. We then discuss when certain extensions of an inside factorial domain are again inside factorial.

Theorem 4.1. *[8, Theorem 4] An integral domain D is inside factorial if and only if $\overline{D}$ is a rational generalized Krull domain with torsion t-class group and $D \subseteq \overline{D}$ is a root extension.*

Proof. ($\Longrightarrow$) Suppose that D is inside factorial. By Theorem 3.1, $D \subseteq D^{\#}$ is a root extension and $D^{\#}$ is a weakly Krull inside factorial domain. By Theorem 3.3 $D^{\#}$ is an AGCD domain. Thus we can apply results from [10] concerning AGCD domains. By [10, Theorem 3.1], $D^{\#} \subseteq \overline{D^{\#}} = \overline{D}$ is a root extension. Hence $D \subseteq \overline{D}$ is a root extension. By [10, Theorem 3.4 and Corollary 3.8] $\overline{D}$ is a PVMD with torsion t-class group. But by Theorem 3.1 again, $\overline{D}$ is weakly Krull. Hence $\overline{D}$ is a generalized Krull domain. It remains to show that for each $P \in X^{(1)}(\overline{D})$, the value group of $\overline{D}_P$ is isomorphic to a subgroup of $(\mathbb{Q}, +)$. Let Q be a Cale basis for $\overline{D}$ with $P = \sqrt{(p)}$ where $p \in Q$. For $0 \neq x \in K$, there is a natural number n with $x^n = \epsilon p^{n_0} q_1^{n_1} \cdots q_s^{n_s}$ where $n_0, \cdots, n_s \in \mathbb{Z}$, $\epsilon \in U(\overline{D})$ and $q_1, \cdots, q_s \in Q$. But $q_1, \cdots, q_s$ are units in $\overline{D}_P$, so we can write $x^n = \epsilon' p^{n_0}$ where ϵ' is a unit in $\overline{D}_P$. If $v \colon K^* \to (\mathbb{R}, +)$ is the valuation for $\overline{D}_P$, then $nv(x) = v(x^n) = v(\epsilon' p^{n_0}) = n_0 v(p)$. Hence $v(x) = \frac{n_0}{n} v(p)$. Thus $\operatorname{im} v$ is order-isomorphic to $(\mathbb{Q}, +)$. ($\Longleftarrow$) [8, Theorem 4] $\square$

We end by considering extensions of inside factorial domains. Let D be inside factorial. Then each overring E of D contained in $\overline{D}$ is again inside factorial (for $D \subseteq E$ is a root extension). Moreover, by Theorem 3.1 if D is weakly Krull, so is E. Next suppose that S is a multiplicatively closed subset of D. Then $D \subseteq \overline{D}$ a root extension gives that $D_S \subseteq \overline{D}_S = \overline{D_S}$ is a root extension, $\overline{D}_S$ is a rational generalized Krull domain, and $\overline{D}_S$ has torsion t-class group since $\overline{D}$ does [2, Theorem 4.4]. Thus D_S is inside factorial. Alternatively, observe that if Q is a Cale basis for D, then $\{q \in Q \mid qD_S \neq D_S\}$ is a Cale basis for D_S. Moreover, if $\varnothing \neq \Lambda \subseteq X^{(1)}(D)$, then $R = \bigcap_{P \in \Lambda} D_P$ is a weakly Krull inside factorial domain. For if we set $S = \left\langle \{q \in Q \mid \sqrt{qD} \notin \Lambda\} \right\rangle$, then $(D^{\#})_S = \left(\bigcap_{P \in X^{(1)}(D)} D_P \right)_S = \bigcap_{P \in \Lambda} D_P = R$.

We next show that $D[X]$ is inside factorial if and only if D is inside factorial and $D[X] \subseteq \overline{D}[X]$ is a root extension. Also see [9, Theorem 3.2]. Now $D[X]$ is inside factorial if and only if $D[X] \subseteq \overline{D}[X]$ is a root extension and $\overline{D}[X]$ is a rational generalized Krull domain with torsion t-class group. But $\overline{D}[X]$ is a rational generalized Krull domain with torsion t-class group if and only if $\overline{D}$ is (for $\overline{D}$ is a generalized Krull domain if and only if $\overline{D}[X]$ is, the natural map $Cl_t(\overline{D}) \to Cl_t(\overline{D}[X])$ is an isomorphism, and for $N \in X^{(1)}(\overline{D}[X])$, $\overline{D}[X]_N$ is a DVR if $N \cap \overline{D} = 0$ and if $N \cap \overline{D} \neq 0$, then $\overline{D}[X]_N = \overline{D}_{N \cap \overline{D}}(X)$ has the same value group as $\overline{D}_{N \cap \overline{D}}$) and $D[X] \subseteq \overline{D}[X]$ a root extension forces $D \subseteq \overline{D}$ to be a root extension. However, $D \subseteq \overline{D}$ a root extension need not imply the $D[X] \subseteq \overline{D}[X]$ is a root extension. Let $D = \mathbb{Z}[\sqrt{5}]$. Then $D \subsetneq \overline{D}$ is a root extension and D is an AGCD (= weakly Krull) inside factorial domain, but $D[X] \subsetneq \overline{D}[X]$ is not a root extension [1, Example 3.6]. Hence $D[X]$ is not inside factorial. According to [1, Theorem 3.4], for an integral domain D, $D[X]$ is an AGCD domain if and only if D is an AGCD domain and $D[X] \subseteq \overline{D}[X]$ is a root extension. Hence if $D[X]$ is inside factorial, $D[X]$ is AGCD (= weakly Krull) if and only if D is.

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